Contents

Preface *vii*

1. **Introduction to 3D Geometry** **1.1–1.23**

Introduction to 3D Coordinate System 1.1
Understanding Coordinates of Point in 3D System 1.2
Direction Angles and Direction Cosines 1.5
Plane Perpendicular to One of the Three Axes 1.7
Line Parallel to One of the Three Axes 1.8
Concept Application Exercise 1.1 *1.9*
Distance Formula 1.10
Distance between Two Points Lying on the Same Axis 1.10
Distance between Two Points Lying on the Line Parallel to One of the Axes 1.10
Distance between Two Points 1.10
Concept Application Exercise 1.2 *1.12*
Section Formula 1.13
Concept Application Exercise 1.3 *1.17*
Sphere 1.18
General Equation of Sphere 1.19
Concept Application Exercise 1.4 *1.21*
Exercises *1.22*
Single Correct Answer Type *1.22*
Answers Key *1.23*

2. **Introduction to Vectors** **2.1–2.64**

Coordinate Axes and Coordinate Planes in Three-Dimensional Space 2.1
Coordinates of a Point in Space 2.1
Distance between Two Points 2.2
Section Formula 2.2
Vector Concept 2.3
Definition 2.4
Position Vector 2.4

Direction Cosines and Direction Ratios 2.4
Types of Vectors 2.5
Addition of Vectors 2.6
Components of a Vector 2.11
Multiplication of a Vector by a Scalar 2.14
Vector Joining Two Points 2.15
Section Formula 2.17
Vector Along the Bisector of Given Two Vectors 2.26
Concept Application Exercise 2.1 2.27
Linear Combination of Vectors 2.29
Linear Combination of Vectors 2.29
Linearly Independent Vectors 2.29
Linearly Dependent Vectors 2.30
Concept Application Exercise 2.2 2.41
Solved Examples 2.42
Exercises 2.55
Single Correct Answer Type 2.55
Multiple Correct Answers Type 2.59
Linked Comprehension Type 2.60
Matrix Match Type 2.62
Numerical Value Type 2.62
Archives 2.63
Answers Key 2.64

3. Different Products of Vectors **3.1–3.107**
Dot (Scalar) Product of Vectors 3.1
Physical Interpretation of Scalar Product 3.1
Geometrical Interpretation of Scalar Product 3.1
Properties of Dot (Scalar) Product 3.2
Applications of Dot (Scalar) Product 3.3
Concept Application Exercise 3.1 3.18
Vector (or Cross) Product of Two Vectors 3.20
Properties of Cross Product 3.21
Physical Interpretation of Cross Product as a Moment of Force 3.21
Geometric Interpretation of Cross Product 3.22
Concept Application Exercise 3.2 3.35
Scalar Triple Product 3.36
Geometrical Interpretation 3.37
Volume of Tetrahedron 3.38
Properties of Scalar Triple Product 3.38

Vector Triple Product 3.46
Lagrange's Identity 3.47
Reciprocal System of Vectors 3.55
Properties 3.55
Concept Application Exercise 3.3 *3.58*
Solved Examples *3.60*
Exercises *3.77*
Single Correct Answer Type *3.77*
Multiple Correct Answers Type *3.87*
Linked Comprehension Type *3.92*
Matrix Match Type *3.96*
Numerical Value Type *3.99*
Archives *3.100*
Answers Key *3.106*

4. Line and Plane **4.1–4.88**
Direction Cosines and Direction Ratios 4.1
Direction Ratios 4.1
Concept Application Exercise 4.1 *4.7*
Line in Space 4.7
Equation of Straight Line Passing through a Given Point and Parallel to a Given Vector 4.7
Equation of Line Passing Through Two Given Points 4.9
Angle between Two Lines 4.14
Point and Line 4.16
Shortest Distance between Two Lines 4.18
Concept Application Exercise 4.2 *4.24*
Plane in Space 4.26
Equation of a Plane in Normal Form 4.26
Equation of a Plane Passing through a Given Point and Normal to a Given Vector 4.28
Equation of a Plane Passing through Three Given Points 4.29
Equation of the Plane that Passes through Point A with Position Vector $\vec{a}$ and is Parallel to Given Vectors $\vec{b}$ and $\vec{c}$ 4.31
Cartesian Form 4.31
Equation of a Plane Passing through a Given Point and Line 4.32
Intercept Form of a Plane 4.33
Equation of a Plane Passing through Two Parallel Lines 4.33
Equation of a Plane Parallel to a Given Plane 4.33
Angle between Two Planes 4.34
Line of Intersection of Two Planes 4.35

Angle between a Line and a Plane 4.40
Equation of a Plane Passing through the Line of Intersection of Two Planes 4.41
Distance of a Point from a Plane 4.43
Distance between Parallel Planes 4.45
Equation of a Plane Bisecting the Angle between Two Planes 4.48
Two Sides of a Plane 4.50
Regular Tetrahedron 4.51
Angle between any Edge and Face not Containing that Edge of Regular Tetrahedron 4.51
Angle between the Two Faces of Regular Tetrahedron 4.51
Height of Regular Tetrahedron 4.52
Concept Application Exercise 4.3 *4.53*
Solved Examples *4.54*
Exercises *4.68*
Single Correct Answer Type *4.68*
Multiple Correct Answers Type *4.77*
Linked Comprehension Type *4.79*
Matrix Match Type *4.80*
Numerical Value Type *4.81*
Archives *4.82*
Answers Key *4.87*

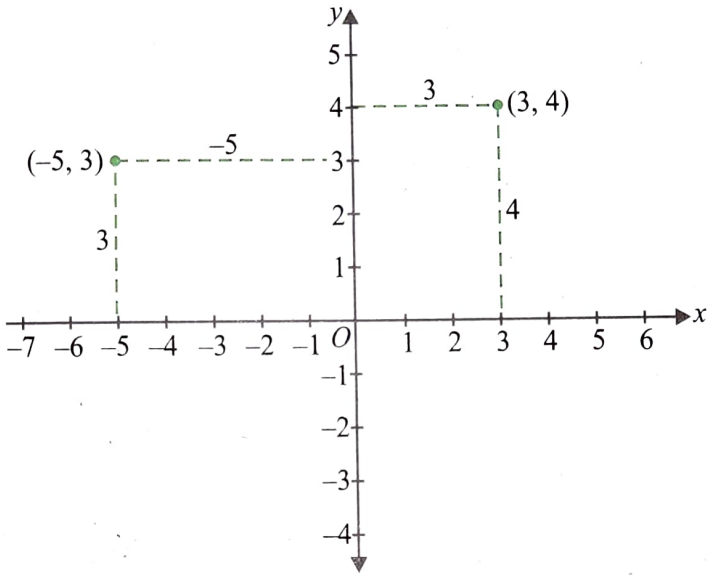
Solutions S.1–S.94
Chapter 1 S.1
Chapter 2 S.6
Chapter 3 S.21
Chapter 4 S.62

Appendix 1:
Chapterwise Solved January 2019 JEE Main Questions (All Sets) A.1–A.3

1 Introduction to 3D Geometry

INTRODUCTION TO 3D COORDINATE SYSTEM

Let us recall the two-dimensional coordinate system. Consider point $P(x, y)$ on the cartesian plane which is formed by two perpendicular axes, x-axis (horizontal axis) and y-axis (vertical axis).

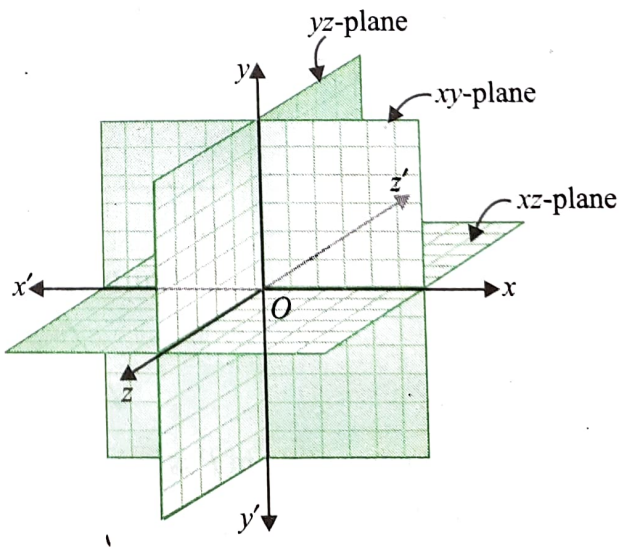


We know that abscissa x of point P is signed distance of P from y-axis i.e., signed distance of P measured parallel to x-axis. Similarly, ordinate y of point P is the signed distance of P from x-axis i.e., signed distance of P measured parallel to y-axis.

Here, two axes divide the plane into four quadrants.

Points on this plane represent ordered pairs of set of cartesian product $R \times R$, where R is set of real numbers.

In three-dimensional coordinate system, we insert one more axis, called z-axis, which goes through the origin of the plane formed by x-axis and y-axis and is perpendicular to both of these axes. Thus, three axes, x-axis, y-axis and z-axis are mutually perpendicular which defines space or three-dimensional (3D) coordinate system. Any point in this coordinate system has coordinates (x, y, z). Points in this system represent ordered triplets of set of cartesian product $R \times R \times R$, where R is set of real numbers. Three axes taken two at a time form three mutually perpendicular planes, xy-plane, yz-plane and zx-plane. These planes divide space in eight regions called **octants**.



Here Ox, Oy and Oz are positive axes and Ox', Oy' and Oz' are corresponding negative axes.

The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants.

Octant / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

ILLUSTRATION 1.1

Name the octants in which the following points lie:

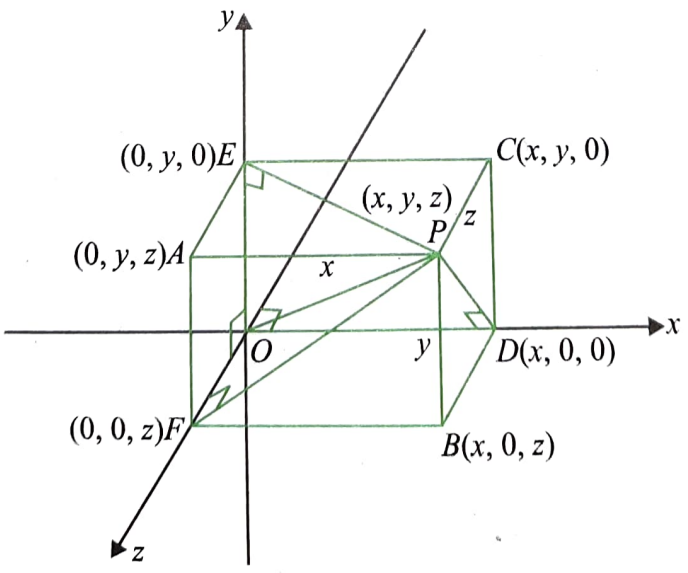
(2, 3, 4), (5, –3, 4), (2, –3, –7), (8, 6, –3), (–3, 5, –8), (–9, 3, 4), (–2, –5, 8), (–7, –8, –1)

Sol.

Point	Octant
(+2, +3, +4)	I
(+5, –3, +4)	IV
(+2, – 3, –7)	VIII
(+8, +6, –3)	V
(–3, +5, –8)	VI
(–9, +3, +4)	II
(–2, –5, +8)	III
(–7, –8, –1)	VII

UNDERSTANDING COORDINATES OF POINT IN 3D SYSTEM

Consider point $P(x, y, z)$ in 3D coordinate system.



The x-coordinates of point P is its signed distance (PA) from yz-plane, i.e., signed distance of P measured parallel to x-axis.

The y-coordinates of point P is its signed distance (PB) from xz-plane, i.e., signed distance of P measured parallel to y-axis.

The z-coordinates of point P is its signed distance (PC) from xy-plane, i.e., signed distance of P measured parallel to z-axis.

Also, from point P, perpendiculars PD, PE and PF are drawn to x-axis, y-axis and z-axis, respectively.

So, $OD = x$ (with sign). Therefore, coordinates of point D are $(x, 0, 0)$.

Also, OD is projection of OP on x-axis.

Similarly, $OE = y$ (with sign) and $OF = z$ (with sign).

So, points E and F have coordinates $(0, y, 0)$ and $(0, 0, z)$, respectively.

OE and OF is projection of OP on y-axis and z-axis, respectively.

Foot of perpendicular from point P on xz-plane is B.

Clearly, $BD = OF = z$ and $BF = OD = x$.

Also, y-coordinate of any point on xz-plane is zero since point has zero distance measured parallel to y-axis.

So, coordinates of point B are $(x, 0, z)$.

Similarly, coordinates of point A (on yz-plane) are $(0, y, z)$ and those of point C(on xy-plane) are $(x, y, 0)$.

Since x-coordinate of all the points lying on yz-plane is zero, we can say that points on yz-plane satisfy equation $x = 0$, which is equation of yz-plane. In other words, locus of point for which $x = 0$ is yz-plane.

Similarly, equation of xz-plane and xy-plane is $y = 0$ and $z = 0$ respectively.

Equations $y = 0$ and $z = 0$ together is equation of x-axis.

Similarly, equation of y-axis is $x = 0$, $z = 0$ together and that of z-axis is $x = 0$, $y = 0$ together.

Distances of Point *P* from Axes

In right angled triangle PBD, $l(PB) = |y|$ and $l(BD) = l(PC) = |z|$.

So,
$$\begin{aligned}(l(PD))^2 &= (l(PB))^2 + (l(PC))^2 \\ &= (|y|)^2 + (|z|)^2 \\ &= y^2 + z^2\end{aligned}$$

$$\therefore \quad l(PD) = \sqrt{y^2 + z^2}$$

This is the distance of point P from x-axis.

Similarly, the distance of point P from y-axis is $l(PE) = \sqrt{x^2 + z^2}$ and that of from z-axis is $l(PF) = \sqrt{x^2 + y^2}$.

Distances of Point *P* from Origin

In right angled triangle ODP,

$$\begin{aligned}(l(OP))^2 &= (l(OD))^2 + (l(DP))^2 \\ &= (|x|)^2 + (\sqrt{y^2 + z^2})^2 \\ &= x^2 + y^2 + z^2\end{aligned}$$

$$\therefore \quad l(OP) = \sqrt{x^2 + y^2 + z^2}$$

ILLUSTRATION 1.2

Let A, B and C be the feet of perpendiculars from a point $P(-4, 5, 9)$ on the x, y, z-axes, respectively. Find the coordinates of A, B and C. Also, if the feet of perpendiculars from P on the xy, yz and zx plane are L, M, N, respectively, then find the coordinates of L, M and N.

Sol. We know that on x-axis, $y, z = 0$, on y-axis, $x, z = 0$ and on z-axis, $x, y = 0$.

Thus, the feet of perpendiculars from given point $P(-4, 5, 9)$ on the axis are $A(-4, 0, 0)$, $B(0, 5, 0)$ and $C(0, 0, 9)$.

We know that, on xy-plane $z = 0$, on yz-plane, $x = 0$ and on zx-plane, $y = 0$.

Thus, the coordinates of feet of perpendicular on the xy, yz and zx plane from the given point are $L(-4, 5, 0)$, $M(0, 5, 9)$ and $N(-4, 0, 9)$.

ILLUSTRATION 1.3

Find the distance of point $P(3, -5, 6)$ from x, y and z-axes and from xy, yz and zx-planes.

Sol. Distance of point $P(3, -5, 6)$ from x-axis

$$= \sqrt{y^2 + z^2} = \sqrt{(-5)^2 + 6^2} = \sqrt{61}$$

Distance of point $P(3, -5, 6)$ from y-axis

$$= \sqrt{x^2 + z^2} = \sqrt{3^2 + 6^2} = \sqrt{45}$$

Distance of point $P(3, -5, 6)$ from z-axis

$$= \sqrt{x^2 + y^2} = \sqrt{3^2 + (-5)^2} = \sqrt{34}$$

Distance of point $P(3, -5, 6)$ from xy-plane $= |z| = 6$

Distance of point $P(3, -5, 6)$ from yz-plane $= |x| = 3$

Distance of point $P(3, -5, 6)$ from zx-plane $= |y| = |-5| = 5$

ILLUSTRATION 1.4

Point $P(2, -3, 4)$ is closest to which coordinate plane and which axis?

Sol. Distance of point $P(2, -3, 4)$ from coordinate planes and coordinate axes is as follows.

	Distance
From xy-plane	$\|z\| = 4$
From yz-plane	$\|x\| = 2$
From zx-plane	$\|y\| = 3$
From x-axis	$\sqrt{y^2 + z^2} = \sqrt{(-3)^2 + 4^2} = 5$
From y-axis	$\sqrt{x^2 + z^2} = \sqrt{2^2 + 4^2} = \sqrt{20}$
From z-axis	$\sqrt{x^2 + y^2} = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

Thus, point P is closest to yz-plane and to z-axis.

ILLUSTRATION 1.5

Find the reflection point $P(2, 3, -4)$ across xy, yz and zx-plane.

Sol. Given point is $P(2, 3, -4)$.

If image of point P in xy-plane is Q then PQ will be parallel to z-axis.

So, x and y-coordinate of point Q will be same and z-coordinate will change its sign.

Hence $Q \equiv (2, 3, 4)$

With similar reasoning, image of P in yz-plane is $R(-2, 3, -4)$ and in zx-plane is $S(2, -3, -4)$.

ILLUSTRATION 1.6

If distance of point P from x, y and z-axis is 2, 3, 6 then find its distance from origin.

Sol. Let $P \equiv (x, y, z)$

Its distance from x, y and z axis is

$$\sqrt{y^2 + z^2} = 2,$$

$$\sqrt{x^2 + z^2} = 3,$$

and $$\sqrt{x^2 + y^2} = 6$$

Squaring and adding, we get $2(x^2 + y^2 + z^2) = (2^2 + 3^2 + 6^2) = 49$

$\therefore$ $$x^2 + y^2 + z^2 = 49/2$$

Hence, $\sqrt{x^2 + y^2 + z^2} = \dfrac{7}{\sqrt{2}}$, which is distance of point P from origin.

DIRECTION ANGLES AND DIRECTION COSINES

In the figure, let OP make positive angles (anticlockwise direction) α, β and γ with the positive x, y and z-axes, respectively.

These angles are known as direction angles.

Taking cosine of these angles we get direction cosines, viz., $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Direction cosines makes it easy to represent the direction of a line and vector in space.

In the figure, in triangle ODP,

$$\cos\alpha = \frac{OD}{OP} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = l$$

Similarly, in triangle OEP,

$$\cos\beta = \frac{OE}{OP} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = m$$

and in triangle OFP,

$$\cos\gamma = \frac{OF}{OP} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = n$$

Thus, $l^2 + m^2 + n^2 = 1$.

This is the relation between the direction cosines.

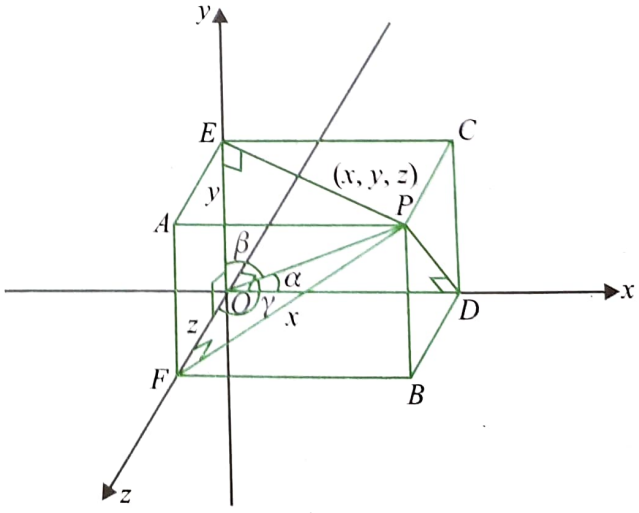


ILLUSTRATION 1.7

If α, β and γ are the angle which a directed line makes with the positive directions of the coordinate axes, then find the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

Sol. The DCs of the line are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

We know that

$$l^2 + m^2 + n^2 = 1$$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \quad 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

ILLUSTRATION 1.8

A line OP through origin O is inclined at 30° and 45° to OX and OY, respectively. Then find the angle at which it is inclined to OZ.

Sol. Let l, m, n be the direction cosines of OP, then $l^2 + m^2 + n^2 = 1$.

If $l = \cos 30° = \frac{\sqrt{3}}{2}$, $m = \cos 45° = \frac{1}{\sqrt{2}}$, then $\frac{3}{4} + \frac{1}{2} + n^2 = 1$

$\Rightarrow \quad n^2 = -\frac{1}{4}$, which is not possible.

So, such a line cannot exist.

ILLUSTRATION 1.9

A line OP makes equal angle with coordinate axes then find its direction cosines.

Sol. Let l, m, n be the direction cosines of OP, which makes equal angle α with coordinate axes.

$$\therefore \quad l = m = n = \cos\alpha$$

Now $$l^2 + m^2 + n^2 = 1$$

$$\therefore \quad 3l^2 = 1$$

$$\Rightarrow \quad l = \pm\frac{1}{\sqrt{3}} = m = n$$

Therefore, $l = m = n = \frac{1}{\sqrt{3}}$ or $l = m = n = -\frac{1}{\sqrt{3}}$

For $l = m = n = \frac{1}{\sqrt{3}}$, point P lies in first octant.

For $l = m = n = -\frac{1}{\sqrt{3}}$, point P lies in seventh octant, where OP makes equal angle $(180° - \alpha)$ with coordinate axes.

ILLUSTRATION 1.10

OP has direction cosines, cos α, cos β and cos γ. Assuming yz-plane as a mirror then what are the direction cosines of the reflected image of OP in the yz-plane.

Sol. Let point P lies in first octant.

$\therefore$ Directional angles of OP are α, β and γ.

If OP is reflected in yz-plane and OP' is reflected line segment then OP' will make same angle β and γ with y-axis and z-axis, respectively, but its inclination with x-axis will be $(180° - \alpha)$

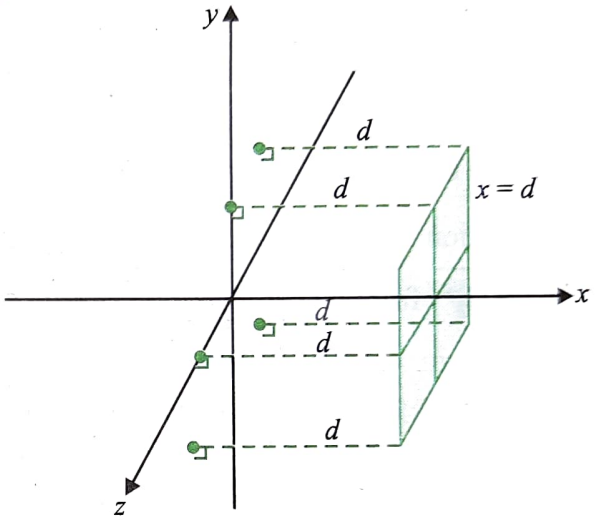
So, direction cosines of OP' are cos $(180° - \alpha)$, cos β, cos γ or $-\cos\alpha$, cos β and cos γ.

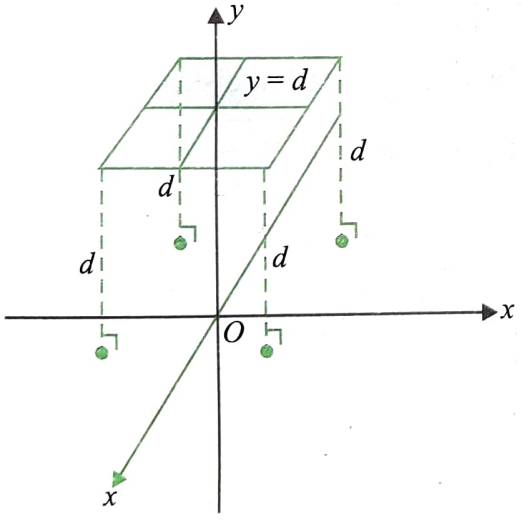
PLANE PERPENDICULAR TO ONE OF THE THREE AXES

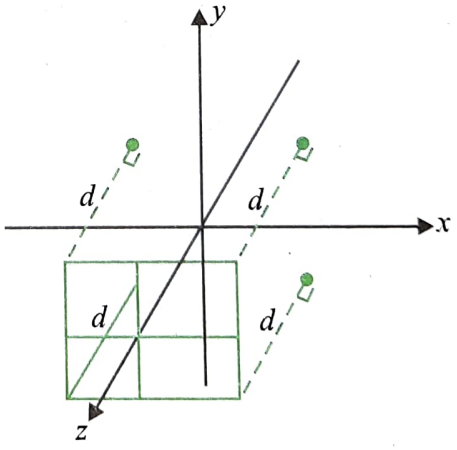
We know that the signed distance of point $P(x_1, y_1, z_1)$ from yz-plane is x_1. If we keep x_1 constant and change the values of y_1 and z_1 arbitrarily, the signed distance of different points thus obtained from yz-plane will remain constant x_1. All these points form a plane parallel to yz-plane.

Thus, locus of the point whose signed distance from yz-plane is constant d is $x = d$. This is equation of plane parallel to yz-plane whose signed distance from yz-plane is d.

Similarly, equation of plane whose signed distance from xz-plane is d is $y = d$ and that of plane whose signed distance from xy-plane d is $z = d$.







Equations of planes at distance d from the yz-plane are $x = \pm d$. Similarly, equations of planes at distance d from xz-plane are $y = \pm d$ and those at distance d from xy-plane are $z = \pm d$.

ILLUSTRATION 1.11

Find the equations of planes passing though point $(-2, 3, -5)$ and which are parallel to

(i) xy-plane (ii) yz-plane (iii) zx-plane

Sol.

(i) Signed distance of point $P(-2, 3, -5)$ from xy-plane is -5.
So, equation of plane through point P and parallel to xy-plane is $z = -5$.

(ii) Signed distance of point $P(-2, 3, -5)$ from xz-plane is 3.
So, equation of plane through point P and parallel to xz-plane is $y = 3$.

(iii) Signed distance of point $P(-2, 3, -5)$ from xy-plane is -2.
So, equation of plane through point P and parallel to xy-plane is $y = -2$.

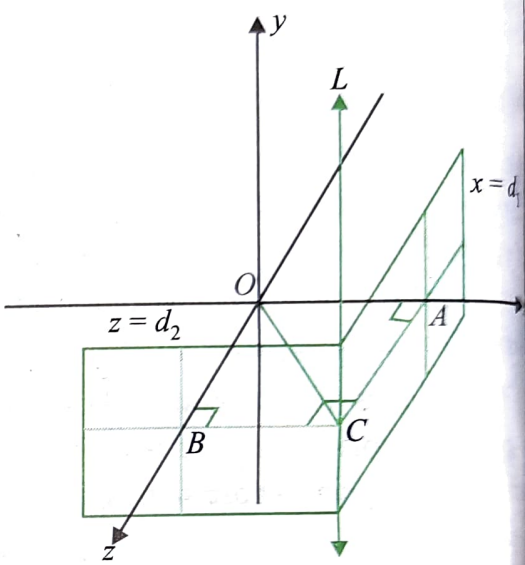
LINE PARALLEL TO ONE OF THE THREE AXES

Two planes in a space intersect on a line.

For example, xy-plane and yz-plane intersect on y-axis whose equation is $x = 0, z = 0$ together.

Now, consider planes $x = d_1$ which is parallel to yz-plane and $z = d_2$, which is parallel to xy-plane.

These two planes intersect on a line L which is parallel to y-axis.



Equation of this line is $x = d_1, z = d_2$ together and lies at distance $\sqrt{d_1^2 + d_2^2}$ from y-axis.

Any point on this line has x-coordinate d_1 and z-coordinate d_2. For different points on the line y-coordinate will be different.

In the figure, $OA = d_1$ and $AC = OB = d_2$.

In triangle OAC, $OC^2 = OA^2 + AC^2 = d_1^2 + d_2^2$

$\therefore \quad OC = \sqrt{d_1^2 + d_2^2}$.

Similarly, $x = d_1, y = d_3$ together is equation of line parallel to z-axis at distance $\sqrt{d_1^2 + d_3^2}$ from it and $y = d_2, z = d_3$ together is equation of line parallel to x-axis at distance $\sqrt{d_2^2 + d_3^2}$ from it.

ILLUSTRATION 1.12

What is the distance of line $x = 3, y = 4$ from axis to which it is parallel?

Sol. Clearly, line $x = 3, y = 4$ is parallel to z-axis.

Its distance from z-axis is $\sqrt{3^2 + 4^2} = 5$

ILLUSTRATION 1.13

Find the equation of line through point $P(2, 3, -4)$, which is

(i) parallel to x-axis (ii) parallel to y-axis (iii) parallel to z-axis.

Sol. Given point is $P(2, 3, -4)$

Now equation of planes through point P which are parallel to yz-plane, xz-plane and xy-plane are $x = 2$, $y = 3$ and $z = -4$, respectively.

(i) equation of line through point P which is parallel to x-axis is $y = 3, z = -4$
(ii) equation of line through point P which is parallel to y-axis is $x = 2, z = -4$
(iii) equation of line through point P which is parallel to z-axis is $x = 2, y = 3$

ILLUSTRATION 1.14

Explain the locus whose equation is $x^2 + z^2 = 25$.

Sol. Let variable point $P(x, y, z)$ satisfy the equation $x^2 + z^2 = 25$.

$\therefore \sqrt{x^2 + z^2} = 5$, i.e., distance of point P from y-axis is constant '5'.

Clearly, all such points lie on the curved surface of cylinder whose axis is y-axis and radius is 5.

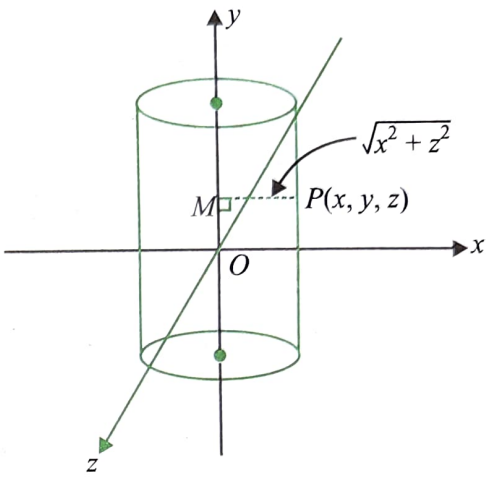


ILLUSTRATION 1.15

Explain equation $y = 3, z = 4$

Sol. Equation $y = 3$ represents plane at distance 3 units on cutting positive side of y-axis.
Equation $z = 4$ represents plane at distance 4 units on cutting positive side of z-axis.
Equations $y = 3, z = 4$ together represents points on the line of intersection of planes $y = 3$ and $z = 4$ which is parallel to x-axis and at distance $\sqrt{3^2 + 4^2} = 5$ from x-axis.

CONCEPT APPLICATION EXERCISE 1.1

1. A parallelepiped is formed by planes drawn through the points $P(6, 8, 10)$ and $Q(3, 4, 8)$ parallel to the coordinate planes. Find the length of an edge of the parallelepiped.
2. What does the equation $x^2 - 2x - 3 = 0$ represent in 3D coordinate system?
3. What is distance of point $P(3, 4, -5)$ from
   (i) plane $z = 6$ (ii) plane $x = 7$
4. Find the image of point $P(-3, 2, 4)$ in xy, yz and zx-plane and also find the octant in which they lie.
5. A line makes angles α, β and γ with the coordinate axes. If $\alpha + \beta = 90°$, then find γ.
6. If a line makes the angles α, β and γ with 3D coordinate axes, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.
7. Let point P be $P(2, 3, 6)$ and O be the origin. Then with which axis the angle of OP is the greatest?

ANSWERS

1. $\sqrt{39}$ **2.** two planes $x = -1$ and $x = 3$ **3.** (i) 11 units (ii) 4 units
5. $\gamma = 90°$ **6.** -1 **7.** x-axis

DISTANCE FORMULA

DISTANCE BETWEEN TWO POINTS LYING ON THE SAME AXIS

Consider two points $P_1(x_1, 0, 0)$ and $P_2(x_2, 0, 0)$ on x-axis.

So, distance between these two points is $P_1P_2 = |x_1 - x_2|$.

Similarly, distance between two points $Q_1(0, y_1, 0)$ and $Q_2(0, y_2, 0)$ on y-axis is $|y_1 - y_2|$ and distance between two points $R_1(0, 0, z_1)$ and $R_2(0, 0, z_2)$ on z-axis is $|z_1 - z_2|$.

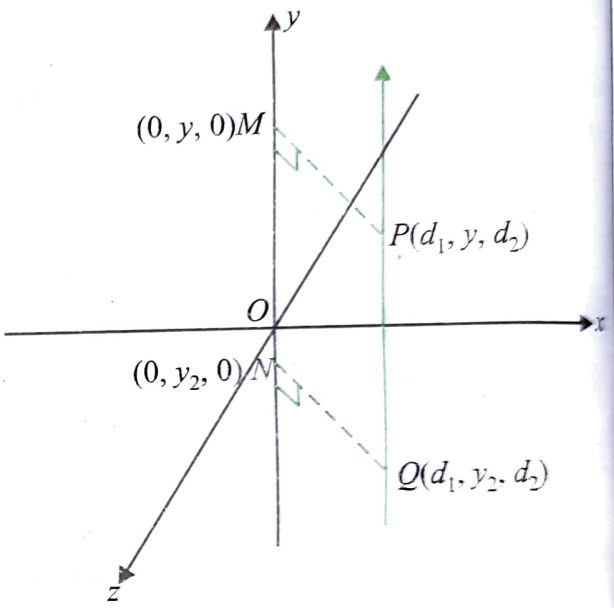
DISTANCE BETWEEN TWO POINTS LYING ON THE LINE PARALLEL TO ONE OF THE AXES

Consider line having equation $x = d_1, z = d_2$. Clearly, this line is parallel to y-axis. Any point on this line has x-coordinate d_1 and z-coordinate d_2. Consider two points on the line $P(d_1, y_1, d_2)$ and $Q(d_1, y_2, d_2)$.

Projection of points P and Q on y-axis are $M(0, y_1, 0)$ and $N(0, y_2, 0)$, respectively.

So, $$PQ = MN = |y_1 - y_2|$$

Similarly, distance between two points on the line parallel to x-axis is difference between their x-coordinates and distance between two points on the line parallel to z-axis is difference between their z-coordinates.



DISTANCE BETWEEN TWO POINTS

Consider two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Cuboid is constructed with PQ as longest diagonal. PA is parallel to x-axis.

Vertex A is foot of perpendicular from vertex Q on side PA. So, coordinates of point A are (x_2, y_1, z_1). Therefore, $PA = |x_1 - x_2|$.

Similarly, $PB = |y_1 - y_2|$ and $PC = |z_1 - z_2|$.

In right angled triangle PAB',

$$PB'^2 = PA^2 + AB'^2$$
$$= PA^2 + PC^2$$
$$= (x_1 - x_2)^2 + (z_1 - z_2)^2$$

Now, in right angled triangle $PB'Q$,

$$PQ^2 = PB'^2 + B'Q^2$$
$$= PB'^2 + PB^2$$
$$= (x_1 - x_2)^2 + (z_1 - z_2)^2 + (y_1 - y_2)^2$$

$$\therefore \quad PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

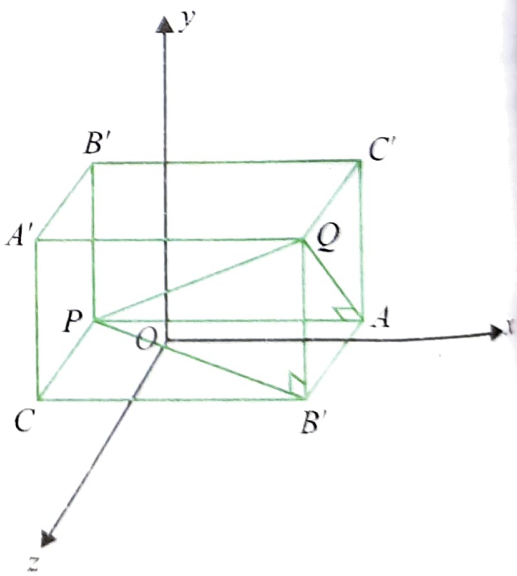


ILLUSTRATION 1.16

Given points $A(-2, 1, 4)$ and $B(5, 0, -7)$. Find the distance between A and B.

Sol. Distance between A and B

$$= \sqrt{(5-(-2))^2 + (0-1)^2 + (-7-4)^2}$$

$$= \sqrt{49+1+121} = \sqrt{171}$$

ILLUSTRATION 1.17

Show that the points $A(1, 0, 2)$, $B(-2, 3, 1)$ and $C(0, 4, -2)$ are the vertices of a right angled triangle.

Sol.

$$AB = \sqrt{(1-(-2))^2 + (0-3)^2 + (2-1)^2} = \sqrt{9+9+1} = \sqrt{19}$$

$$AC = \sqrt{(1-0)^2 + (0-4)^2 + (2-(-2))^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$BC = \sqrt{(-2-0)^2 + (3-4)^2 + (1-(-2))^2} = \sqrt{4+1+9} = \sqrt{14}$$

Clearly, $AC^2 = AB^2 + BC^2$

So, triangle ABC is right angled triangle with right angle at vertex B.

ILLUSTRATION 1.18

Show that points $A(2, 3, 5)$, $B(6, 1, 6)$ and $C(14, -3, 8)$ are collinear.

Sol. If A, B and C lie on the same line, then the largest distance between any two of the three points will be equal to the sum of the two smaller distances.

Now,

$$AB = \sqrt{(6-2)^2 + (1-3)^2 + (6-5)^2}$$

$$= \sqrt{4^2+2^2+1^2} = \sqrt{21}$$

$$AC = \sqrt{(14-2)^2 + (-3-3)^2 + (8-5)^2}$$

$$= \sqrt{12^2 + (-6)^2 + 3^2} = \sqrt{189} = 3\sqrt{21}$$

$$BC = \sqrt{(14-6)^2 + (-3-1)^2 + (8-6)^2}$$

$$= \sqrt{8^2 + (-4)^2 + 2^2} = \sqrt{84} = 2\sqrt{21}$$

Hence, $AB + BC = AC$.

Therefore, point A, B and C are collinear with point B dividing AC in the ratio $1 : 2$.

ILLUSTRATION 1.19

Find the coordinates of a point on z-axis which are at a distance of 7 units from the point $P(3, -2, 5)$.

Sol. Point on the z-axis is $A(0, 0, z)$.

Given point is $P(3, -2, 5)$.

Also, it is given that $AP = 7$

$\therefore \quad AP^2 = 50$

$\Rightarrow \quad (3-0)^2 + (-2-0)^2 + (5-z)^2 = 49$

$\Rightarrow \quad 9 + 4 + (5-z)^2 = 49$

$\Rightarrow \quad (5-z)^2 = 36$

$\Rightarrow \quad 5 - z = \pm 6$

$\Rightarrow \quad z = -1 \text{ or } 11$

Thus, the coordinates of the required points are $(0, 0, -1)$ and $(0, 0, 11)$.

ILLUSTRATION 1.20

Given points $A(-2, 1, 4)$ and $B(5, 0, -7)$. Find all the points which are equidistant from A and B. Also, find the locus of these points.

Sol. Let point $P(x, y, z)$ be equidistant from points A and B.

So, $PA = PB$

$\therefore \quad \sqrt{(x-(-2))^2 + (y-1)^2 (z-4)^2} = \sqrt{(x-5)^2 + (y-0)^2 + (z-(-7))^2}$

$\Rightarrow \quad (x+2)^2 + (y-1)^2 + (z-4)^2 = (x-5)^2 + y^2 + (z+7)^2.$

$\Rightarrow \quad x^2 + 4x + 4 + y^2 - 2y + 1 + z^2 - 8z + 16 = x^2 - 10x + 25 + y^2 + z^2 + 14z + 49.$

$\Rightarrow \quad 14x - 2y - 22z = 53.$

Thus, all points which are equidistant from A and B satisfy the equation $14x - 2y - 22z = 53$.

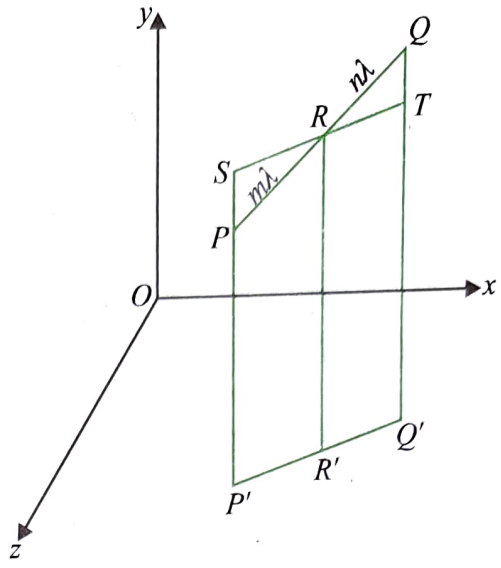
CONCEPT APPLICATION EXERCISE 1.2

1. Verify the following:
   (i) $(0, 7, -10)$, $(1, 6, -6)$ and $(4, 9, -6)$ are the vertices of an isosceles triangle.
   (ii) $(-1, 2, 1)$, $(1, -2, -5)$, $(4, -7, 8)$ and $(2, -3, 4)$ are the vertices of a parallelogram.
2. Distance of point P from origin is 7 units, from x-axis is $\sqrt{13}$ units and from y-axis is $\sqrt{40}$ units. Then find coordinates of point P.
3. Let $x^2 + 3y^2 = 3$ be the equation of an ellipse in the xy-plane. If $A(-\sqrt{3}, 0, 0)$ and $B(-\sqrt{3}, 0, 2)$ are two points, then find point P on the ellipse such that $\angle APB = 45°$.
4. Explain the locus of point $P(3, y, z)$ whose distance of origin is 5.

ANSWERS

2. $(\pm 6, \pm 3, \pm 2)$ **3.** $(0, \pm 1, 0)$

SECTION FORMULA



Consider two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let us find the coordinates of point R which divides PQ in the ratio $m : n$ internally.

Draw perpendiculars PP', QQ' and RR' to the xz-plane. P', Q' and R' are feet of these perpendiculars on xz-plane.

Clearly $PP' = y_1$, $QQ' = y_2$.

Let $RR' = y$.

Now through point R draw a line ST parallel to the line $P'Q'$. Line ST will intersect the line PP' externally at the point S and the line QQ' at T.

Clearly, triangles PSR and QTR are similar.

$$\therefore \quad \frac{m\lambda}{n\lambda} = \frac{PR}{QR} = \frac{SP}{QT} = \frac{SP' - PP'}{QQ' - TQ'} = \frac{RR' - PP'}{QQ' - RR'} = \frac{y - y_1}{y_2 - y}$$

Simplifying, we get,

$$y = \frac{my_2 + ny_1}{m + n}$$

Similarly, by drawing perpendiculars to the yz-plane and xy plane, we get

$$x = \frac{mx_2 + nx_1}{m + n} \text{ and } z = \frac{mz_2 + nz_1}{m + n}$$

Hence, the coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m : n$ are

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

If the point R divides PQ externally in the ratio $m : n$, then its coordinates are obtained by replacing n by $-n$ so that coordinates of point R will be

$$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$$

- The coordinates of the mid-point of the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.
- The coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

ILLUSTRATION 1.21

Find the coordinates of the point which divides the line segment joining the points (–4, 2, 3) and (3, –5, 7) in the ratio

(i) 3 : 4 internally, and (ii) 3 : 4 externally.

Sol.

(i) Let $R(x, y, z)$ be the point that divides the line segment joining points $P(-4, 2, 3)$ and $Q(3, -5, 7)$ internally in the ratio 3 : 4. Then

$$x = \frac{3(3)+3(-4)}{3+4},\ y = \frac{3(-5)+4(2)}{3+4} \text{ and } z = \frac{3(7)+4(3)}{3+4}$$

i.e., $$x = -\frac{3}{7},\ y = -1,\ z = \frac{33}{7}$$

Thus, $R \equiv \left(-\frac{3}{7}, -1, \frac{33}{7}\right)$.

(ii) Let $S(x, y, z)$ divides PQ externally in the ratio 3 : 4 then

$$x = \frac{3(3)-3(-4)}{3-4},\ y = \frac{3(-5)-4(2)}{3-4} \text{ and } z = \frac{3(7)-4(3)}{3-4}$$

i.e., $$x = -21,\ y = 23,\ z = -9$$

Thus, $S \equiv (-21, 23, -9)$.

ILLUSTRATION 1.22

Prove that points $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divides PR.

Sol. Let point $Q(5, 4, -6)$ divide the line segment joining points $P(3, 2, -4)$ and $R(9, 8, -10)$ in the ratio $k : 1$.

Therefore, by section formula,

$$(5, 4, -6) \equiv \left(\frac{k(9)+3}{k+1}, \frac{k(8)+2}{k+1}, \frac{k(-10)-4}{k+1}\right)$$

If points P, Q and R are collinear, we must find same value of k from each comparison.

Comparing x-coordinates, we have

$$\frac{9k+3}{k+1} = 5$$

$$\Rightarrow \quad 9k + 3 = 5k + 5$$

$$\Rightarrow \quad 4k = 2 \Rightarrow k = \frac{2}{4} = \frac{1}{2}$$

Comparing y-coordinates, we have

$$\frac{8k+2}{k+1} = 4$$

$$\Rightarrow \quad k + 1 = 2k + 2$$

$$\Rightarrow \quad k = \frac{1}{2}$$

Similarly, comparing z-coordinates we get $k = \frac{1}{2}$.

Thus, points P, Q and R are collinear.

Also, Q divides PR in the ratio 1 : 2.

ILLUSTRATION 1.23

Consider two points $P(-2, 3, 5)$ and $Q(1, -4, 3)$. Find the point where PQ meets
(i) yz-plane (ii) xz-plane and (iii) xy-plane

Sol. Let point R divides joint of $P(-2, 3, 5)$ an $Q(1, -4, 3)$ in the ratio $\lambda_1 : 1$.
Hence, by section formula, the coordinates of point of R_1 are

$$R \equiv \left(\frac{\lambda_1(1) - 2}{\lambda_1 + 1}, \frac{\lambda_1(-4) + 3}{\lambda_1 + 1}, \frac{\lambda_1(3) + 5}{\lambda_1 + 1} \right) \quad (1)$$

(i) If point R lies on yz-plane then x-coordinate of R will be zero.

$$\therefore \quad \frac{\lambda_1(1) - 2}{\lambda_1 + 1} = 0 \text{ or } \lambda_1 = 2$$

Putting this value of λ_1 in (1), we get

$$R \equiv \left(0, \frac{-5}{3}, \frac{11}{3} \right)$$

(ii) If point R lies on xz-plane then y-coordinate of R will be zero.

$$\therefore \quad \frac{\lambda_1(-4) + 3}{\lambda_1 + 1} = 0 \text{ or } \lambda_1 = \frac{3}{4}$$

Putting this value of λ_1 in (1), we get

$$R \equiv \left(\frac{-5}{7}, 0, \frac{29}{7} \right)$$

(iii) If point R lies on xy-plane then z-coordinate of R will be zero.

$$\therefore \quad \frac{\lambda_1(3) + 5}{\lambda_1 + 1} = 0 \text{ or } \lambda_1 = \frac{5}{3}$$

Putting this value of λ_1 in (1), we get

$$R \equiv \left(\frac{11}{2}, -\frac{29}{2}, 0 \right)$$

ILLUSTRATION 1.24

Prove that line joining $A(1, -2, -3)$ and $B(-3, 4, 6)$ intersect x-axis. Also, find the point on the x-axis.

Sol. Let line joining A and B intersect x-axis at $C(x, 0, 0)$.
Let $AC : CB = k : 1$

$$\therefore \quad C \equiv \left(\frac{-3k+1}{k+1}, \frac{4k-2}{k+1}, \frac{6k-3}{k+1} \right) \equiv (x, 0, 0)$$

$$4k - 2 = 0 \text{ and } 6k - 3 = 0$$

So, from both the equations $k = \frac{1}{2}$.

We would not have got the same value of k if line had not met x-axis. Thus, line AB meets x-axis.

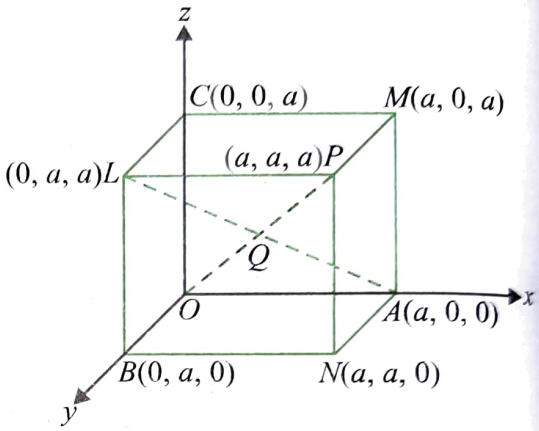
Also, $C \equiv \frac{-1}{3}, 0, 0$

ILLUSTRATION 1.25

Prove that three diagonals of a cube intersect. Also find the acute angle between any two diagonals.

Sol. Cube with side length 'a' and having three of its sides along positive coordinate axes is as shown in the figure.

Three diagonals are OP, BE and FC.



Midpoint of OP has coordinates $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.

Also, midpoint of BE and FC has coordinates $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.

Thus, diagonals are concurrent at $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right) \equiv M$.

Let us find angle between diagonals OP and AD.

Consider triangle MPD.

$$OP = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}a$$

$$\therefore \quad MP = MD = \frac{\sqrt{3}a}{2}$$

Also, $PD = \sqrt{2}a$

Now in triangle MPD, using cosine rule,

$$\cos\theta = \frac{(MP)^2 + (MD)^2 - (PD)^2}{2 \times MP \times MD} = \frac{\frac{3}{4}a^2 + \frac{3}{4}a^2 - 2a^2}{2 \times \frac{\sqrt{3}a}{2} \times \frac{\sqrt{3}a}{2}}$$

$$= \frac{-\frac{1}{2}a^2}{\frac{3a^2}{2}} = -\frac{1}{3}$$

$$\therefore \quad \theta = \cos^{-1}\left(-\frac{1}{3}\right) = \pi - \cos^{-1}\frac{1}{3}$$

Thus, acute angle between diagonals is $\cos^{-1}\frac{1}{3}$.

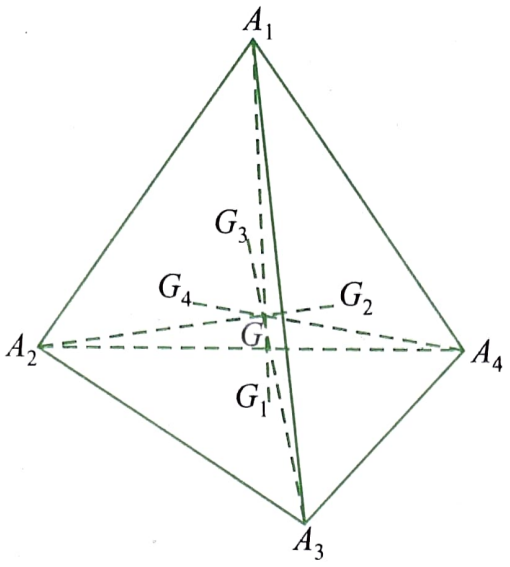
ILLUSTRATION 1.26

Prove that the line segments joining vertex to the centroid of the opposite face of tetrahedron are concurrent.

Sol. Consider tetrahedron $A_1A_2A_3A_4$ with $A_i \equiv (x_i, y_i, z_i)$; $i = 1, 2, 3, 4$.

Centroid of triangle $A_2A_3A_4$ is

$$G_1 \equiv \left(\frac{x_2 + x_3 + x_4}{3}, \frac{y_2 + y_3 + y_4}{3}, \frac{z_2 + z_3 + z_4}{3}\right)$$



Now, we have point G on A_1G_1 which divides it in the ratio 3 : 1.

$$\therefore \quad G \equiv \left(\frac{3\left(\frac{x_2+x_3+x_4}{3}\right)+1(x_1)}{3+1}, \frac{3\left(\frac{y_2+y_3+y_4}{3}\right)+1(y_1)}{3+1}, \frac{3\left(\frac{z_2+z_3+z_4}{3}\right)+1(z_1)}{3+1} \right)$$

$$\equiv \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4} \right)$$

Centroid of triangle $A_1A_3A_4$ is $G_2 \equiv \left(\frac{x_1+x_3+x_4}{3}, \frac{y_1+y_3+y_4}{3}, \frac{z_1+z_3+z_4}{3} \right)$.

Now, we have point on A_2G_2 which divides it in the ratio 3 : 1, having coordinates

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4} \right) \equiv G$$

Thus, A_1G_1 and A_2G_2 intersect at G.

Similarly, A_3G_3 and A_4G_4 pass through point G, where G_3 and G_4 are centroids of triangular faces opposite to vertices A_3 and A_4, respectively.

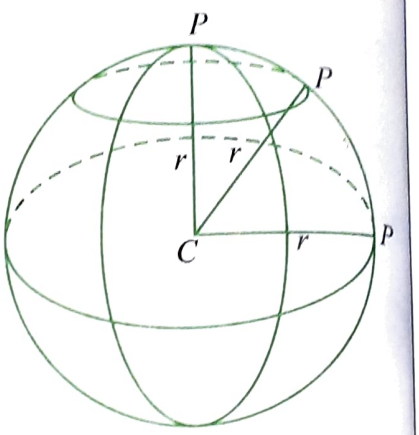
Concept Application Exercise 1.3

1. Using section formula, prove that points $A(3, 2, -4)$, $B(5, 4, -6)$ and $C(9, 8, -10)$ are collinear.
2. Find the coordinates of the points which trisect the line segment joining the points $P(4, 2, -6)$ and $Q(10, -16, 6)$.
3. Equation $ax + by + cz + d = 0$ represents the equation of plane in the space. Prove that points $A(2, 3, -4)$ and $B(7, 2, 0)$ lie on the same side of the plane $x - y + 2z - 7 = 0$.
4. Prove that line segments joining midpoints of opposite edges of tetrahedron are concurrent.

Answers

2. $(8, -10, 2)$

SPHERE



In 3D coordinate system, a sphere is a locus of variable point $P(x, y, z)$ whose distance from fixed point $C(x_0, y_0, z_0)$ is constant r. The fixed point is called centre and constant distance r is called radius of the sphere.

Thus, $CP = r$ or $CP^2 = r^2$

So, $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$, which is equation of the sphere.

ILLUSTRATION 1.27

Find the equation of a sphere with centre $(2, -1, 3)$ and radius 4.

Sol. The equation of sphere is given by

$$(x - 2)^2 + (y - (-1))^2 + (z - 3)^2 = 4^2,$$

which simplifies to $x^2 + y^2 + z^2 - 4x + 2y - 6z = 2$

ILLUSTRATION 1.28

Find the equation of a sphere on which the two points $A(4, 1, -1)$ and $B(6, 7, 9)$ lie directly opposite one another (that is, the line through them forms a diameter of the sphere.

Sol. Clearly the centre is the midpoint of A and B, which is

$$C \equiv \left(\frac{4+6}{2}, \frac{1+7}{2}, \frac{-1+9}{2}\right) \equiv (5, 4, 4).$$

Radius of the sphere is

$$CA = \sqrt{(5-4)^2 + (4-1)^2 + (4-(-1))^2}$$

$$= \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}$$

$\therefore$ Equation of sphere is $(x - 5)^2 + (y - 4)^2 + (z - 4)^2 = 35$, which simplifies to

$$x^2 + y^2 + z^2 - 10x - 8y - 8z + 22 = 0$$

ILLUSTRATION 1.29

Find the equation of the largest possible sphere with centre $(3, 2, 4)$ that is fully contained within the first octant. (tangentially "touching", a coordinate plane is permissible).

Sol. Clearly the largest possible sphere is that which is touching all the coordinate planes.

Since the y-coordinate of 2 is the smallest of the three coordinates and is 2 units from the xz-coordinate plane, this will be the radius.

Thus, equation of the sphere will be $(x - 4)^2 + (y - 2)^2 + (z - 5)^2 = 4$, which simplifies to

$$x^2 + y^2 + z^2 - 8x - 4y - 10z + 41 = 0$$

ILLUSTRATION 1.30

The sphere $(x+6)^2+(y-1)^2+(z-4)^2=100$ intersects the yz-coordinate plane, forming a circle. What is the radius of this circle?

Sol. The points where sphere intersect yz–plane, $x=0$.

So, putting $x=0$ in the given equation of sphere we have

$$((0)+6)^2+(y-1)^2+(z-4)^2=100$$

or $$(y-1)^2+(z-4)^2=64$$

Clearly this is equation of circle in yz–plane.

The centre of the circle is $(1, 4)$ and radius is '8'.

GENERAL EQUATION OF SPHERE

From the previous discussion we get that equation of sphere takes the form

$x^2+y^2+z^2+2gx+2fy+2hz+c=0$, which can put in centre–radius form as

$(x+g)^2+(y+f)^2+(z+h)^2=g^2+f^2+h^2-c$

Centre of the sphere is $(-g, -f, -h)$ and radius is $\sqrt{g^2+f^2+h^2-c}$

ILLUSTRATION 1.31

Find the centre and radius of the sphere $x^2+y^2+z^2-2x+6y-4z=22$.

Sol. The given equation can be put in the form

$$(x-1)^2+(y+3)^2+(z-2)^2=36$$

Thus, the sphere has a centre of $(-1, 3, -2)$ and a radius of $r=6$.

ILLUSTRATION 1.32

Find the intercept of sphere $(x+1)^2+(y-4)^2+(z-6)^2=33$ on z–axis.

Sol. When sphere cuts x–axis, $x=0$ and $y=0$.

So, from the equation of sphere, we have

$$((0)+1)^2+((0)-4)^2+(z-6)^2=33$$

$\Rightarrow$ $$1^2+(-4)^2+(z-6)^2=33$$

$\Rightarrow$ $$(z-6)^2=16$$

$\Rightarrow$ $$z-6=\pm 4$$

$\Rightarrow$ $$z=10 \text{ or } 2$$

Thus sphere intersect z–axis at $A(0, 0, 2)$ and $B(0, 0, 10)$.

So, $AB=8$

ILLUSTRATION 1.33

Find the equation of the sphere which passes through $(1, 0, 0)$ $(0, 1, 0)$, $(0, 0, 1)$ and has radius as small as possible.

Sol. Let the equation of the required sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad (1)$$

As (1) passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1), we get $1 + 2u + d = 0$, $1 + 2v + d = 0$, $1 + 2w + d = 0$

$$\Rightarrow \quad u = v = w = -\frac{1}{2}(d + 1)$$

If R is the radius of (1), then $R^2 = u^2 + v^2 + w^2 - d$

$$\Rightarrow \quad R^2 = \frac{3}{4}(d+1)^2 - d$$

$$= \frac{3}{4}\left[d^2 + 2d + 1 - \frac{4}{3}d\right]$$

$$= \frac{3}{4}\left[d^2 + \frac{2}{3}d + 1\right]$$

$$= \frac{3}{4}[(d + 1/3)^2 + 1 - 1/9] = \frac{3}{4}[(d + 1/3)^2 + 8/9]$$

The last equation shows that R^2 (and thus R) will be least if and only if $d = -1/3$.

Therefore, $u = v = w = -\frac{1}{2}\left(1 - \frac{1}{3}\right) - \frac{1}{3}$

Hence, the equation of the required sphere is $x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) - \frac{1}{3} = 0$

or $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$

ILLUSTRATION 1.34

A sphere of constant radius k passes through the origin and meet the axes in A, B and C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

Sol. Let equation of any sphere passing through the origin and having radius k be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \quad (1)$$

As the radius of the sphere is k, we get

$$u^2 + v^2 + w^2 = k^2 \quad (2)$$

Note that (1) meets x-axis in $O(0, 0, 0)$ and $A(-2u, 0, 0)$; y-axis in $O(0, 0, 0)$ and $B(0, -2v, 0)$ and z-axis in $O(0, 0, 0)$ and $C(0, 0, -2w)$.

Suppose the centroid of the triangle ABC be (α, β, γ).

Then $\quad \alpha = -\frac{2u}{3}, \beta = -\frac{2v}{3}, \gamma = -\frac{2w}{3}$

$$\Rightarrow \quad u = -\frac{3\alpha}{2}, v = -\frac{3\beta}{2}, w = \frac{3v}{2}$$

Putting this in (2), we get $\left(\frac{-3}{2}\alpha\right)^2 + \left(\frac{-3}{2}\beta\right)^2 + \left(\frac{-3}{2}\gamma\right)^2 = k^2$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 = \frac{4}{9}k^2.$$

This show that centroid of the triangle ABC lies on $x^2 + y^2 + z^2 = \frac{4}{9}k^2$.

Concept Application Exercise 1.4

1. Find the equation of the sphere whose centre is (3, 1, 2) and radius is 5.
2. Find the equation of the sphere passing through (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).
3. Find the locus of a point which moves such that the sum of the squares of its distance from the points $A(1, 2, 3)$, $B(2, -3, 5)$ and $C(0, 7, 4)$ is 120.
4. A point $P(x, y, z)$ is such that $3PA = 2PB$ where A and B are the points (1, 3, 4) and (1, −2, −1) respectively. Find the equation to the locus of the point P and verify that the locus is a sphere. Find its centre and radius.

Answers

1. $x^2 + y^2 + z^2 - 6x - 2y - 4z - 11 = 0$
2. $x^2 + y^2 + z^2 - x - y - z = 0$
3. $x^2 + y^2 + z^2 - 2x - 4y - 8z - 1 = 0$
4. $x^2 + y^2 = z^2 - 2x - 14y - 16z + 42 = 0$; Centre ≡ (1, 7, 8); Radius = $6\sqrt{2}$

Exercises

Single Correct Answer Type

1. The distance of point $P(3, 4, 5)$ from the xy-plane is
   (1) 3 units (2) 4 units (3) 5 units (4) 550
2. The length of foot of perpendicular drawn from the point $P(-3, -4, 5)$ on y-axis
   (1) $\sqrt{41}$ (2) $\sqrt{34}$ (3) 5 (4) none of these
3. M is the foot of the perpendicular drawn from a point $P(-1, 7, -5)$ on the xy-plane. The coordinates of point M are
   (1) $(-1, 7, 0)$ (2) $(1, -7, 0)$ (3) $(7, 5, 0)$ (4) $(-1, 6, 0)$
4. If feet of perpendiculars from point $P(1, 2, 3)$ on x, y and z-axis are A, B and C respectively, then distance of centroid of triangle ABC from origin is
   (1) $\frac{\sqrt{14}}{3}$ (2) $\frac{\sqrt{14}}{2}$ (3) $\sqrt{14}$ (4) $3\sqrt{14}$
5. If feet of perpendiculars from point $P(1, 2, 3)$ on xy, yz and zx-plane are A, B and C respectively, then distance of centroid of triangle ABC from origin is
   (1) $\frac{\sqrt{14}}{3}$ (2) $\frac{2\sqrt{14}}{3}$ (3) $\frac{3\sqrt{14}}{2}$ (4) $3\sqrt{14}$
6. A point R with x–coordinate 6 lies on the line segment joining the points $P(2, -3, 4)$ and $Q(8, 0, 10)$. Then distance of R from origin is
   (1) $\sqrt{47}$ (2) $\sqrt{41}$ (3) $\sqrt{53}$ (4) $3\sqrt{7}$
7. The volume of the cuboid having combined equations of opposite faces as $x^2 - 3x - 4 = 0$, $y^2 - 5y + 6 = 0$ and $z^2 + 6z + 8 = 0$ is
   (1) 20 cu. units (2) 10 cu. units (3) 15 cu. units (4) 24 cu. units
8. Number of spheres touching all the coordinate planes and having fixed radius is
   (1) 2 (2) 4 (3) 6 (4) 8
9. If the sum of the squares of the distance of a point from the three coordinate axes be 36, then its distance from the origin is
   (1) 5 (2) $3\sqrt{2}$ (3) $\sqrt{21}$ (4) $3\sqrt{3}$
10. The xy-plane divides the line joining the points $(-1, 3, 4)$ and $(2, -5, 6)$
    (1) internally in the ratio 2 : 3 (2) externally in the ratio 2 : 3
    (3) internally in the ratio 3 : 2 (4) externally in the ratio 3 : 2
11. If the points $(-1, 3, 2)$, $(-4, 2, -2)$ and $(5, 5, \lambda)$ are collinear, then $\lambda =$
    (1) 10 (2) –10 (3) –8 (4) 8
12. From which of the following the distance of the point $(1, 2, 3)$ is $\sqrt{10}$?
    (1) origin (2) x-axis (3) y-axis (4) z-axis
13. Distance of the point $(1, 2, 3)$ from the coordinate axes are
    (1) 13, 10, 5 (2) $\frac{1}{\sqrt{13}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{5}}$ (3) $\sqrt{5}, \sqrt{13}, \sqrt{10}$ (4) $\sqrt{13}, \sqrt{10}, \sqrt{5}$

14. If the centroid of triangle whose vertices are $(a, 1, 3)$, $(-2, b, -5)$ and $(4, 7, c)$ be the origin, then the value of a, b and c are
(1) $-2, -8, -2$ (2) $2, 8, -2$ (3) $-2, -8, -2$ (4) $7, -1, 0$

15. The point of intersection of the line joining the points $(3, 4, 1)$ and $(5, 1, 6)$ and the xy-plane is
(1) $(13, 23, 0)$ (2) $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$ (3) $(-13, 23, 0)$ (4) $\left(-\frac{13}{5}, \frac{23}{5}, 0\right)$

16. The coordinates of a point which is equidistance from the points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ are given by
(1) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (2) $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ (3) $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$ (4) $\left(-\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right)$

17. The coordinates of a point P are $(3, 12, 4)$ with respect to origin O, then the direction cosines of OP are
(1) $3, 12, 4$ (2) $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ (3) $\frac{3}{\sqrt{13}}, \frac{1}{\sqrt{13}}, \frac{2}{\sqrt{13}}$ (4) $\frac{3}{13}, \frac{12}{13}, \frac{4}{13}$

18. A line joining point P with the origin makes angles of $45°$ and $60°$ with the positive axes of X and Y respectively. The angle made by the same line with the positive axis of Z, is
(1) $30°$ or $60°$ (2) $60°$ or $90°$ (3) $90°$ or $120°$ (4) $60°$ or $120°$

19. The points $A(5, -1, 1)$, $B(7, -4, 7)$, $C(1, -6, 10)$ and $D(-1, -3, 4)$ are vertices of a
(1) square (2) rhombus (3) rectangle (4) None of these

20. Which of the following point lies inside the sphere $x^2 + y^2 - 4x - 6y + 8y - 7 = 0$?
(1) $(1, -2, 1)$ (2) $(0, 0, 3)$ (3) $(0, 1, 2)$ (4) $(1, 2, 0)$

21. Number of points common to the spheres $x^2 + y^2 + z^2 = 36$ and $x^2 + y^2 + z^2 - 20x + 84 = 0$ is
(1) zero (2) one (3) infinite (4) none of these

22. The xy-plane cuts the sphere $x^2 + y^2 + z^2 - 2x - 4y + 6z - 4 = 0$ in a circle, the radius of which is
(1) 2 (2) 3 (3) 4 (4) 5

23. If equation $6x + 3y - 2z = 18$ represents the equation of plane in the space then the centroid of triangle ABC where A, B, C are the points of intersection of the planes with the coordinate axes are
(1) $(1, 2, -3)$ (2) $(-1, 2, 3)$ (3) $(-1, -2, -3)$ (4) $(1, -2, 3)$

Answers Key

EXERCISES

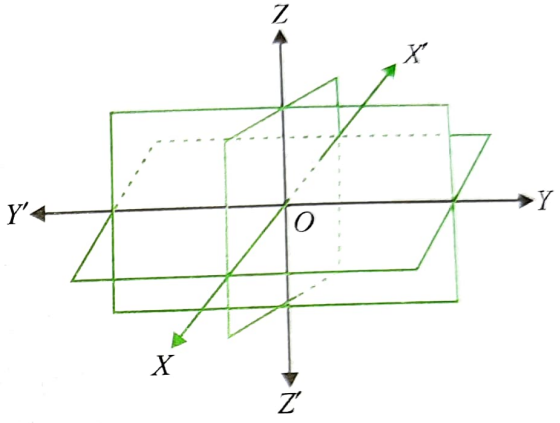
Single Correct Answer Type

1. (3)	**2.** (2)	**3.** (1)	**4.** (1)	**5.** (2)	**6.** (3)	**7.** (2)	**8.** (4)	**9.** (2)	**10.** (2)
11. (1)	**12.** (3)	**13.** (4)	**14.** (3)	**15.** (2)	**16.** (1)	**17.** (4)	**18.** (4)	**19.** (2)	**20.** (4)
21. (2)	**22.** (2)	**23.** (1)							

2 Introduction to Vectors

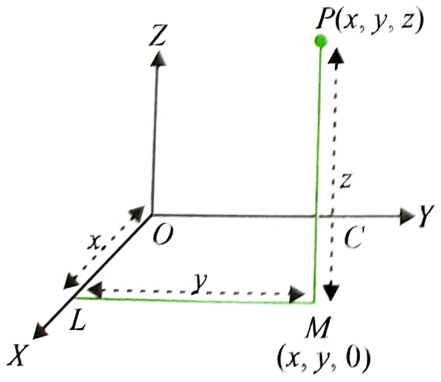
COORDINATE AXES AND COORDINATE PLANES IN THREE-DIMENSIONAL SPACE

Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other as shown in the following figure.

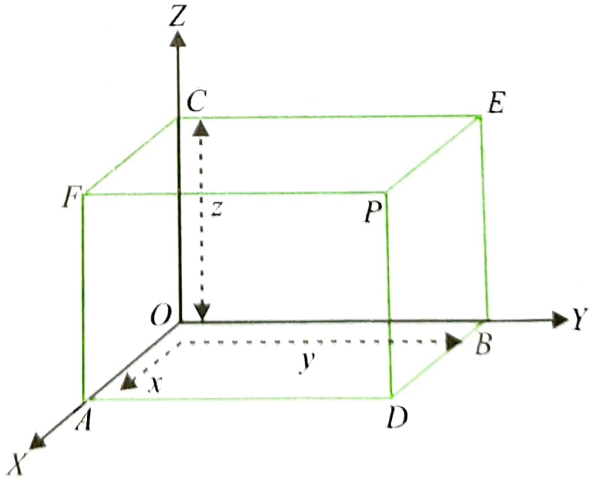


These three planes intersect along the lines $X'OX$, $Y'OY$ and $Z'OZ$, called the x-, y- and z-axes, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the *rectangular coordinate system*. The planes XOY, YOZ and ZOX, called respectively, the XY-plane, the YZ-plane and the ZX-plane, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line $Z'OZ$ as perpendicular to the plane XOY. If the plane of the paper is considered to be horizontal, then the line $Z'OZ$ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ are taken as negative. Similarly, the distances measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY' as negative, in front of the YZ-plane along OX as positive and to the back of it along OX' as negative. The point O is called the *origin* of the coordinate system. The three coordinate planes divide the space into eight parts known as *octants*. These octants can be named as $XOYZ$, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$ and $XOY'Z'$ and are denoted by I, II, III, IV, V, VI, VII and VIII, respectively.

COORDINATES OF A POINT IN SPACE



(i)



(ii)

Consider a point P in space, we drop a perpendicular PM on the XY-plane with M as the foot of this perpendicular. Then, from point M, we draw a perpendicular ML to the x-axis, meeting it at L. Let OL be x, LM be y and MP be z. Then x, y and z are called the x-, y- and z-coordinates, respectively, of point P in the space. In the figure, we may note that the point $P(x, y, z)$ lies in the octant $XOYZ$ and so all x, y, z are positive. If P was in any other octant, the signs of x, y and z would change accordingly. Thus, to each point P in the space, there corresponds an ordered triplet (x, y, z) of real numbers.

We observe that if $P(x, y, z)$ is any point in the space, then x, y and z are perpendicular distances from YZ, ZX and XY planes, respectively.

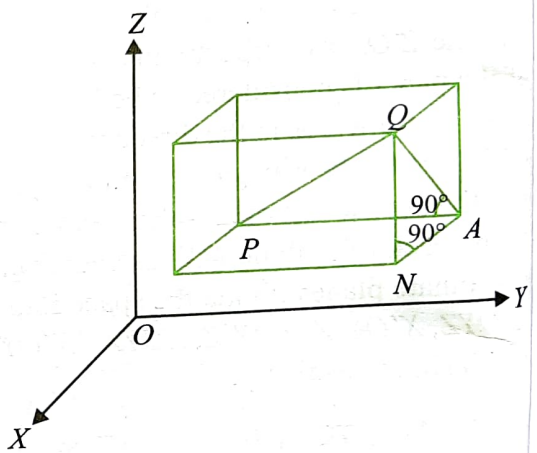
Note: The coordinates of the origin O are $(0, 0, 0)$. The coordinates of any point on the x-axis will be $(x, 0, 0)$ and the coordinates of any point in the YZ-plane will be $(0, y, z)$.

The sign of the coordinates of a point determines the octant in which the point lies. The following table shows the signs of the coordinates in the eight octants:

Octant Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

DISTANCE BETWEEN TWO POINTS

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelepiped with one diagonal PQ.



Now, since $\angle PAQ$ is a right angle, it follows that in triangle PAQ,

$$PQ^2 = PA^2 + AQ^2 \quad (1)$$

Also, triangle ANQ is right-angled with $\angle ANQ$ being the right angle. Therefore,

$$AQ^2 = AN^2 + NQ^2 \quad (2)$$

From (1) and (2), we have

$$PQ^2 = PA^2 + AN^2 + NQ^2$$

Now $PA = y_2 - y_1$, $AN = x_2 - x_1$ and $NQ = z_2 - z_1$

Hence, $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$\therefore$ $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

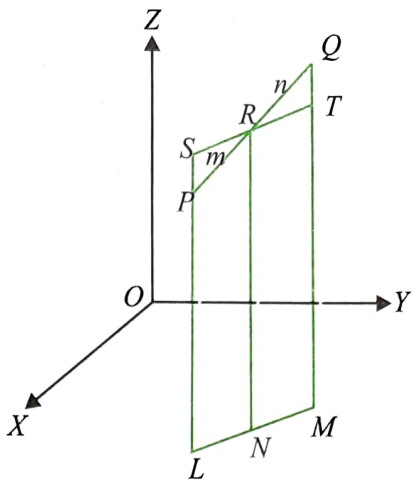
This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is origin O, then $OQ = \sqrt{x_2^2 + y_2^2 + z_2^2}$, which gives the distance between the origin O and any point $Q(x_2, y_2, z_2)$.

SECTION FORMULA

Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Let point $R(x, y, z)$ divide PQ in the given ratio $m : n$ internally. Draw PL, QM and RN perpendicular to the XY-plane. Obviously $PL \parallel RN \parallel OM$ and the feet

of these perpendiculars lie in the XY-plane. Through point R draw a line ST parallel to line LM. Line ST will intersect line LP externally at point S and line MQ at T, as shown in the figure.



Also note that quadrilaterals $LNRS$ and $NMTR$ are parallelograms.

The triangles PSR and QTR are similar. Therefore,

$$\frac{m}{n}=\frac{PR}{QR}=\frac{SP}{QT}=\frac{SL-PL}{QM-TM}=\frac{NR-PL}{QM-NR}=\frac{z-z_1}{z_2-z}$$

$$\Rightarrow \quad z=\frac{mz_2+nz_1}{m+n}$$

Hence, the coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m : n$ are $\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}$ and $\frac{mz_2+nz_1}{m+n}$.

If point R divides PQ externally in the ratio $m : n$, then its coordinates are obtained by replacing n with $-n$ so that the coordinates become $\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}$ and $\frac{mz_2-nz_1}{m-n}$.

Note:

- If R is the midpoint of PQ, then $m : n = 1:1$; so $x=\frac{x_1+x_2}{2}, y=\frac{y_1+y_2}{2}, z=\frac{z_1+z_2}{2}$.

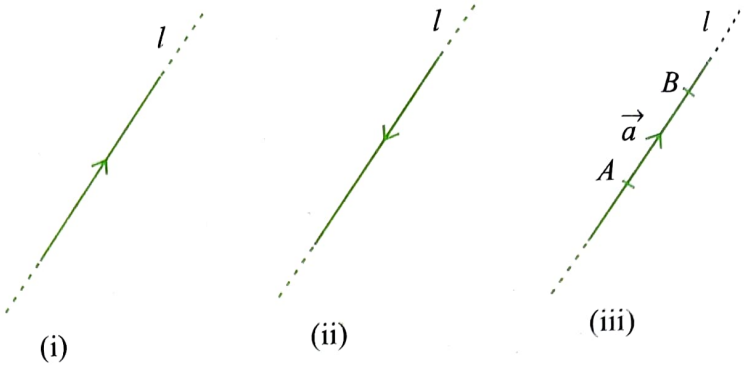
  These are the coordinates of the midpoint of the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.
- The coordinates of the point R which divides PQ in the ratio $k : 1$ are obtained by taking $k=\frac{m}{n}$, which are given by $\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$.
- If vertices of triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, and $AB = c$, $BC = a$, $AC = b$, then centroid of the triangle is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$ and its incenter is $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right)$

VECTOR CONCEPT

In our day-to-day life, we come across many queries such as 'What is your height?' and 'How should a football player hit the ball to give a pass to another player of his team?' Observe that a possible answer to the first query may be 1.5 m, a quantity that involves only one value (magnitude) which is a real number. Such quantities are called *scalars*. However, an answer to the second query is a quantity (called force) which involves muscular strength (magnitude) and direction (in which another player is positioned). Such quantities are called *vectors*. In mathematics, physics and engineering, we frequently come across with both types of quantities, namely scalar quantities such as length, mass, time, distance, speed, area, volume, temperature, work, money, voltage, density and resistance and vector quantities such as displacement, velocity, acceleration, force, momentum and electric field intensity.

Let 'l' be a straight line in a plane or a three-dimensional space. This line can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a directed line [Fig. (i), (ii)].

Now observe that if we restrict the line l to the line segment AB, then a magnitude is prescribed on line (i) with one of the two directions, so that we obtain a directed line segment, Fig. (iii). Thus, a directed line segment has magnitude as well as direction.



(i) (ii) (iii)

DEFINITION

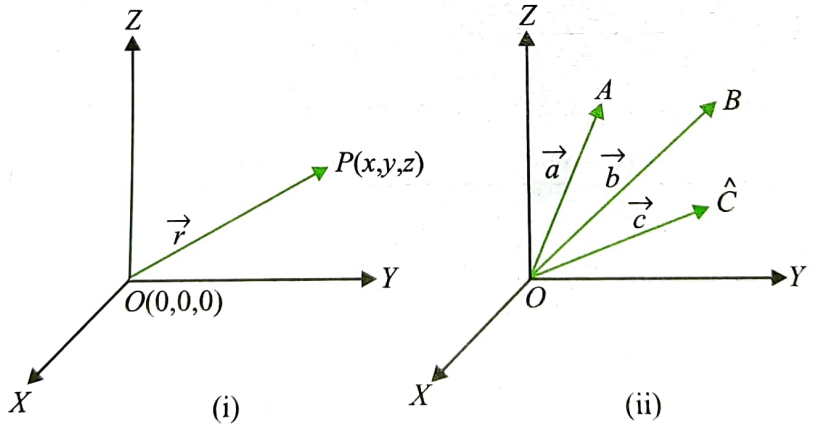
A quantity that has magnitude as well as direction is called a vector.

Notice that a directed line segment is a vector [Fig. (iii)], denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$, and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

Point A from where vector $\overrightarrow{AB}$ starts is called its initial point, and point B where it ends is called its terminal point. The distance between initial and terminal points of a vector is called the magnitude (or length) of the vector, denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a. The arrow indicates the direction of the vector.

POSITION VECTOR

Consider a point P in space having coordinates (x, y, z) with respect to the origin O $(0, 0, 0)$. Then, the vector $\overrightarrow{OP}$ having O and P as its initial and terminal points, respectively, is called the position vector of the point P with respect to O. Using the distance formula, the magnitude of $\overrightarrow{OP}$ (or $\vec{r}$) is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.



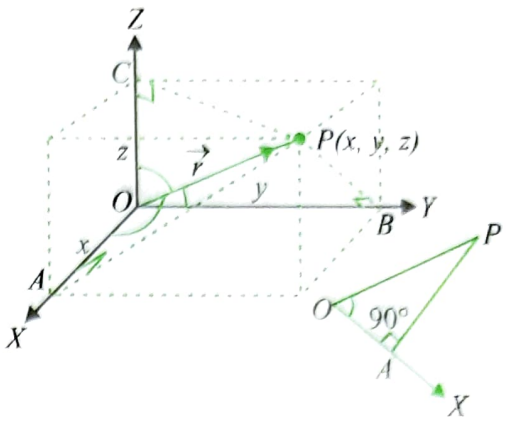
(i) (ii)

In practice, the position vectors of points A, B, C, etc., with respect to origin O are denoted by $\vec{a}, \vec{b}, \vec{c}$, etc., respectively [Fig. (ii)].

DIRECTION COSINES AND DIRECTION RATIOS

Consider the position vector $\overrightarrow{OP}$ (or $\vec{r}$) of a point P (x, y, z). The angles α, β and γ made by the vector $\vec{r}$ with the positive directions of x-, y- and z-axes, respectively, are called its direction angles. The cosine values of these angles, i.e., $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, are called direction cosines of the vector $\vec{r}$ and are usually denoted by l, m and n, respectively.

From the figure, one may note that triangle OAP is right angled, and in it, we have $\cos\alpha = x/r$ (r stands for $|\vec{r}|$). Similarly, from the right-angled triangles OBP and OCP, we may write $\cos\beta = y/r$ and $\cos\gamma = z/r$. Thus, the coordinates of point P may also be expressed as (lr, mr, nr). The numbers lr, mr and nr, proportional to the direction cosines, are called the direction ratios of vector $\vec{r}$ and are denoted by a, b and c, respectively.



TYPES OF VECTORS

Zero Vector

A vector whose initial and terminal points coincide is called a zero vector (or null vector) and is denoted as $\vec{0}$. A zero vector cannot be assigned a definite direction as it has zero magnitude or, alternatively, it may be regarded as having any direction. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vector.

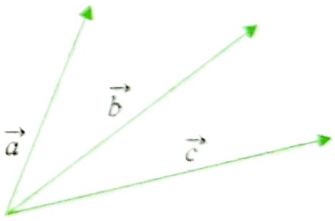
Unit Vector

A vector of unit magnitude is called a unit vector. Unit vectors are denoted by small letters with a cap on them.

Thus, $\hat{a}$ is unit vector of $\vec{a}$, where $|\hat{a}| = 1$, i.e., if vector $\vec{a}$ is divided by magnitude $|\vec{a}|$, then we get a unit vector in the direction of $\vec{a}$. Thus, $\dfrac{\vec{a}}{|\vec{a}|} = \hat{a} \Leftrightarrow \vec{a} = |\vec{a}|\hat{a}$, where $\hat{a}$ is the unit vector in the direction of $\vec{a}$.

Coinitial Vectors

Two or more vectors having the same initial point are called coinitial vectors.



Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and direction regardless of the positions of their initial points. They are written as $\vec{a} = \vec{b}$.

Negative of a Vector

A vector whose magnitude is the same as that of a given vector (say, $\overrightarrow{AB}$), but whose direction is opposite to that of it, is called negative of the given vector. For example, vector $\overrightarrow{BA}$ is negative of vector $\overrightarrow{AB}$ and is written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Free Vectors

Vectors whose initial points are not specified are called free vectors.

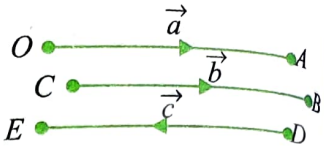
Localised Vectors

A vector drawn parallel to a given vector, but through a specified point as the initial point, is called a localised vector.

Parallel Vectors

Two or more vectors are said to be parallel if they have the same support or parallel support.

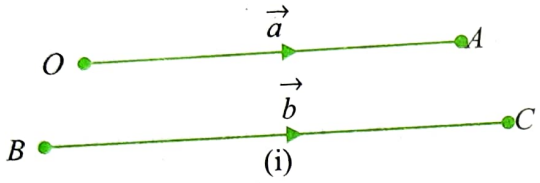
Parallel vectors may have equal or unequal magnitudes and their directions may be same or opposite as shown in the figure.

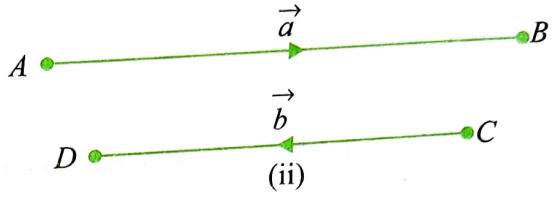


Like and Unlike Vectors

Two parallel vectors having the same direction are called *like vectors* [see Fig. (i)].

Two parallel vectors having opposite directions are called *unlike vectors* [see Fig. (ii)].



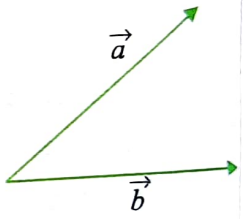


Collinear Vectors

Vectors $\vec{a}$ and $\vec{b}$ are collinear if they have same direction or are parallel or anti-parallel. Since their magnitudes are different, we can find some scalar λ for which $\vec{a} = \lambda \vec{b}$. If $\lambda > 0$, $\vec{a}$ and $\vec{b}$ are in the same direction; if $\lambda < 0$, $\vec{a}$ and $\vec{b}$ are in the opposite directions. Collinear vectors are often called dependent vectors.

Non-collinear Vectors

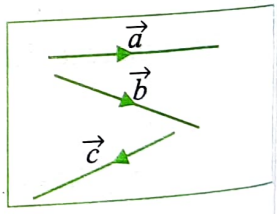
Two vectors acting in different directions are called non-collinear vectors. Non-collinear vectors are often called independent vectors. Here we cannot write vector $\vec{a}$ in terms of $\vec{b}$, though they have the same magnitude. However, we can find component of one vector in the direction of the other. Two non-collinear vectors describe plane.



Coplanar Vectors

Two parallel vectors or non-collinear vectors are always coplanar or two vectors $\vec{a}$ and $\vec{b}$ in different directions determine unique plane in space. Now if vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. Generally more than two vectors are coplanar if all are in the same plane.

Three non-coplanar vectors describe space.

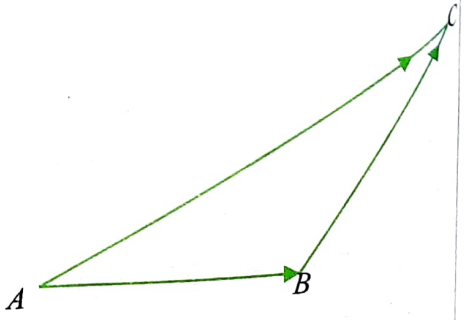


ADDITION OF VECTORS

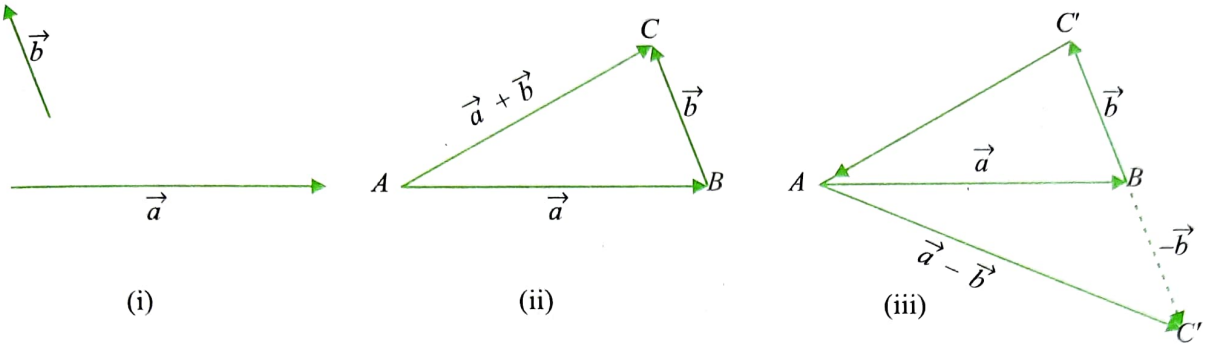
A vector $\overrightarrow{AB}$ simply means the displacement from point A to point B. Now consider a situation where a boy moves from A to B and then from B to C. The net displacement made by the boy from point A to point C is given by vector $\overrightarrow{AC}$ and is expressed as

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

This is known as the triangle law of vector addition.



In general, if we have two vectors $\vec{a}$ and $\vec{b}$ [Fig. (i)], then to add them, they are positioned such that the initial point of one coincides with the terminal point of the other [Fig. (ii)].



(i) (ii) (iii)

For example, in Fig. (ii), we have shifted vector $\vec{b}$ without changing its magnitude and direction so that its initial point coincides with the terminal point of $\vec{a}$. Then the vector $\vec{a} + \vec{b}$, represented by the third side AC of the triangle ABC, gives us the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$, i.e., in triangle ABC [Fig. (ii)], we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Since $\overrightarrow{AC} = -\overrightarrow{CA}$, from the above equation, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$$

This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided [Fig. (iii)].

Now, construct a vector $\overrightarrow{BC}$ so that its magnitude is same as that of vector $\overrightarrow{BC}$, but the direction is opposite to that of $\overrightarrow{BC}$ [Fig. (iii)], i.e.,

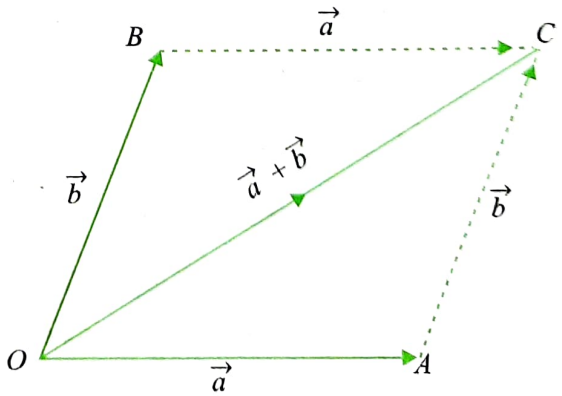
$$\overrightarrow{BC} = -\overrightarrow{BC'}$$

Then, on applying triangle law from Fig. (iii), we have

$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'} = \overrightarrow{AB} + (-\overrightarrow{BC}) = \vec{a} - \vec{b}$$

Vector $\overrightarrow{AC'}$ is said to represent the difference of $\vec{a}$ and $\vec{b}$.

Now, consider a boat going from one bank of a river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors—one is the velocity imparted to the boat by its engine and the other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat actually starts travelling with a different velocity. To have a precise idea about the effective speed and direction (i.e., the resultant velocity) of the boat, we have the following law of vector addition.



If we have two vectors $\vec{a}$ and $\vec{b}$ represented by the two adjacent sides of a parallelogram in magnitude and direction (see figure), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point. This is known as the parallelogram law of vector addition.

Note:

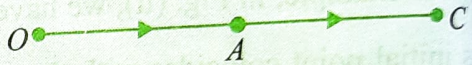
- From figure, using the triangle law, one may note that

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC} \qquad (\because \overrightarrow{AC} = \overrightarrow{OB})$$

or $$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$

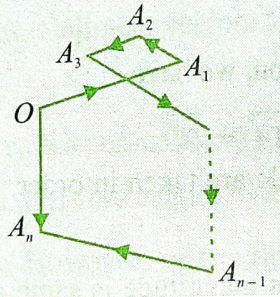
which is the parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

- If $\overrightarrow{OA}$ and $\overrightarrow{AC}$ are collinear, their sum is still $\overrightarrow{OC}$. Although in this case we do not have a triangle or a parallelogram in their usual sense.



- As from the figure:

$\overrightarrow{OA} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{OA_n}$ by the polygon law of addition.



Properties of Vector Addition

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative property)
2. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (associative property)
3. $\vec{a} + \vec{0} = \vec{a}$ (additive identity)
4. $\vec{a} + (-\vec{a}) = \vec{0}$ (additive inverse)
5. $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ and $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$

ILLUSTRATION 2.1

If vector $\vec{a} + \vec{b}$ bisects the angle between $\vec{a}$ and $\vec{b}$, then prove that $|\vec{a}| = |\vec{b}|$.

Sol. We know that vector $\vec{a} + \vec{b}$ is along the diagonal of the parallelogram whose adjacent sides are vectors $\vec{a}$ and $\vec{b}$. Now if $\vec{a} + \vec{b}$ bisects the angle between vectors $\vec{a}$ and $\vec{b}$, then the parallelogram must be a rhombus; hence, $|\vec{a}| = |\vec{b}|$.

ILLUSTRATION 2.2

If $\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{BO}+\overrightarrow{OC}$, then prove that B is the midpoint of AC.

Sol. $\overrightarrow{AO}+\overrightarrow{OB}=\overrightarrow{BO}+\overrightarrow{OC}$

$\Rightarrow \quad \overrightarrow{AB}=\overrightarrow{BC}$

Thus, vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear

$\Rightarrow$ Points A, B, C are collinear

Also $|\overrightarrow{AB}|=|\overrightarrow{BC}|$

$\Rightarrow$ B is the midpoint of AC

ILLUSTRATION 2.3

$ABCDE$ is a pentagon. Prove that the resultant of forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.

Sol. $\vec{R}=\overrightarrow{AB}+\overrightarrow{AE}+\overrightarrow{BC}+\overrightarrow{DC}+\overrightarrow{ED}+\overrightarrow{AC}$

$=(\overrightarrow{AB}+\overrightarrow{BC})+(\overrightarrow{AE}+\overrightarrow{ED}+\overrightarrow{DC})+\overrightarrow{AC}$

$=\overrightarrow{AC}+\overrightarrow{AC}+\overrightarrow{AC}=3\overrightarrow{AC}$

ILLUSTRATION 2.4

Prove that the resultant of two forces acting at point O and represented by $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is given by $2\overrightarrow{OD}$, where D is the midpoint of BC.

Sol. $\vec{R}=\overrightarrow{OB}+\overrightarrow{OC}$

$=(\overrightarrow{OD}+\overrightarrow{DB})+(\overrightarrow{OD}+\overrightarrow{DC})$

$=2\overrightarrow{OD}+(\overrightarrow{DB}+\overrightarrow{DC})=2\overrightarrow{OD}+\vec{0}=2\overrightarrow{OD}$

(Since D is the midpoint of BC, we have $\overrightarrow{DB}=-\overrightarrow{DC}$)

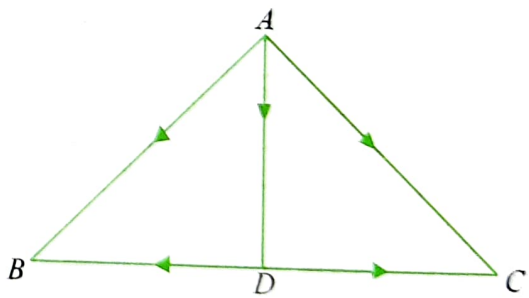


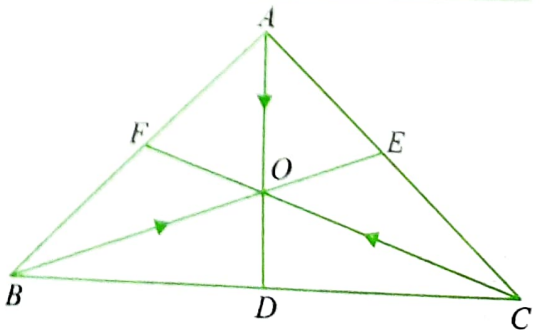
ILLUSTRATION 2.5

Prove that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

Sol. $\overrightarrow{AB}+\overrightarrow{AC}=2\overrightarrow{AD}$

$\overrightarrow{BC}+\overrightarrow{BA}=2\overrightarrow{BE}$

$\overrightarrow{CA}+\overrightarrow{CB}=2\overrightarrow{CF}$



Adding, we get

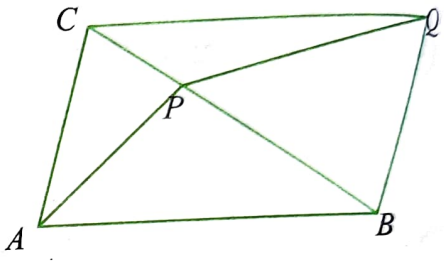
$$(\overrightarrow{AB} + \overrightarrow{BA}) + (\overrightarrow{AC} + \overrightarrow{CA}) + (\overrightarrow{BC} + \overrightarrow{CB}) = 2(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF})$$

or $$\vec{0} + \vec{0} + \vec{0} = 2(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF})$$

or $$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$$

ILLUSTRATION 2.6

ABC is a triangle and P any point on BC. If $\overrightarrow{PQ}$ is the sum of $\overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PC}$, show that $ABQC$ is a parallelogram and Q, therefore, is a fixed point.



Sol. Here $$\overrightarrow{PQ} = \overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PC}$$

$$\overrightarrow{PQ} - \overrightarrow{PC} = \overrightarrow{AP} + \overrightarrow{PB}$$

$$\overrightarrow{PQ} + \overrightarrow{CP} = \overrightarrow{AP} + \overrightarrow{PB}$$

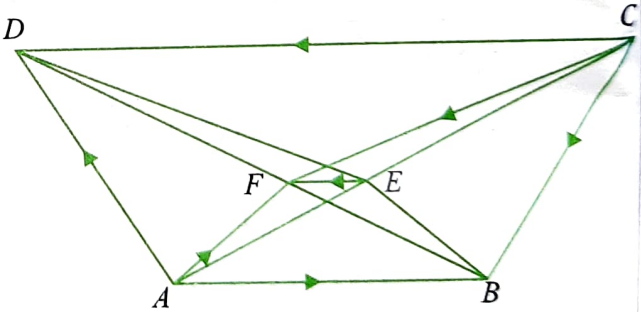
$$\overrightarrow{CQ} = \overrightarrow{AB} \Rightarrow CQ = AB \text{ and } CQ \parallel AB$$

Therefore, $ABQC$ is a parallelogram.
But A, B and C are given to be fixed points and $ABQC$ is a parallelogram. Therefore, Q is a fixed point.

ILLUSTRATION 2.7

Two forces $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are acting at the vertex A of a quadrilateral $ABCD$ and two forces $\overrightarrow{CB}$ and $\overrightarrow{CD}$ at C. Prove that their resultant is given by $4\overrightarrow{EF}$, where E and F are the midpoints of AC and BD, respectively.

Sol. $\overrightarrow{AB} + \overrightarrow{AD} = 2\overrightarrow{AF}$, where F is the midpoint of BD.



$$\overrightarrow{CB} + \overrightarrow{CD} = 2\overrightarrow{CF}$$

$$\therefore \quad \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 2(\overrightarrow{AF} + \overrightarrow{CF})$$

$$= -2(\overrightarrow{FA} + \overrightarrow{FC})$$

$$= -2[2\overrightarrow{FE}], \text{ where } E \text{ is the midpoint of } AC$$

$$= 4\overrightarrow{EF}$$

ILLUSTRATION 2.8

If $O(\vec{0})$ is the circumcentre and O' the orthocentre of a triangle ABC, then prove that

i. $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OO'}$

ii. $\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{O'O}$

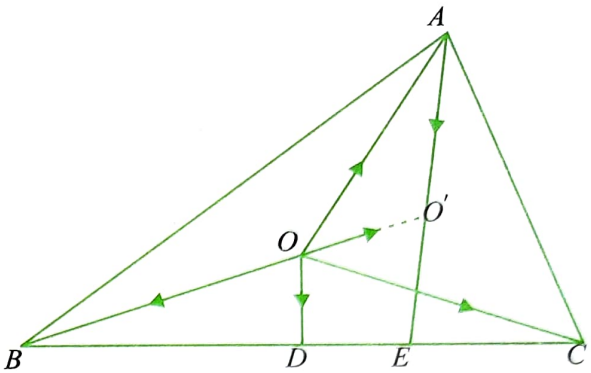
iii. $\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{AO} = \overrightarrow{AP}$

where AP is the diameter through A of the circumcircle.

Sol. O is the circumcentre, which is the intersection of the right bisectors of the sides of the triangle, and O' is the orthocenter, which is the point of intersection of altitudes drawn from the vertices. Also, from geometry, we know that

$$2OD = AO'.$$

$$\therefore \quad 2\overrightarrow{OD} = \overrightarrow{AO'} \qquad \text{(i)}$$



i. To prove: $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OO'}$

Now $\overrightarrow{OB} + \overrightarrow{OC} = 2\overrightarrow{OD} = \overrightarrow{AO'}$

$\Rightarrow \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AO'} = \overrightarrow{OO'}$ [by (i)]

ii. To prove: $\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{OO'}$

L.H.S. $= 2\overrightarrow{DO} + 2\overrightarrow{O'D}$ [by (i)]

$= 2(\overrightarrow{O'D} + \overrightarrow{DO}) = 2\overrightarrow{O'O}$

iii. To prove: $\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{AO} = \overrightarrow{AP}$

L.H.S. $= 2\overrightarrow{AO'} - \overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C}$

$= 2\overrightarrow{AO'} + (\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C}$

$= 2\overrightarrow{AO'} + 2\overrightarrow{O'O} = 2\overrightarrow{AO}$

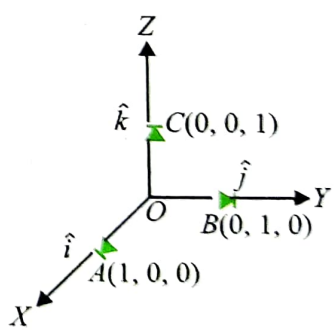
$= \overrightarrow{AP}$ (where AP is the diameter through A of the circumcircle).

COMPONENTS OF A VECTOR

Let us take the points $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$ on the x-axis, y-axis and z-axis, respectively.

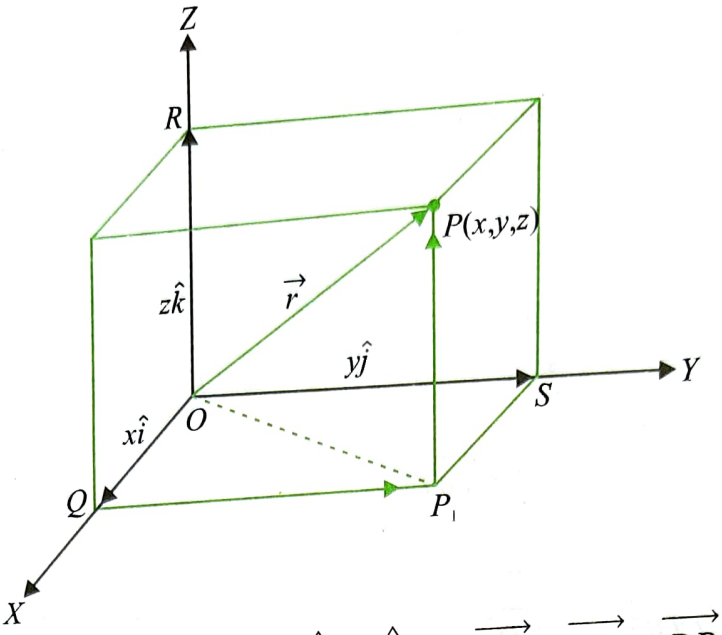
Then, clearly $|\overrightarrow{OA}| = 1, |\overrightarrow{OB}| = 1$ and $|\overrightarrow{OB}| = 1$

The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, each having magnitude 1, are called unit vectors along the axes OX, OY and OZ, respectively, and are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively.



Now, consider the position vector $\overrightarrow{OP}$ of a point $P(x, y, z)$ as shown in the figure. Let P_1 be the foot of the perpendicular from P on the plane XOY.

We, thus, see that P_1P is parallel to the z-axis. As $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the x-, y- and z-axes, respectively, and by the definition of the coordinates of P, we have $\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}$. Similarly, $\overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{OQ} = x\hat{i}$.



Therefore, it follows that $\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{QP_1} = x\hat{i} + y\hat{j}$ and $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the position vector of P with reference to O is given by

$$|\overrightarrow{OP}| (\text{or } \vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

This form of any vector is called its component form. Here, x, y and z are called the scalar components of $\vec{r}$, and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the vector components of $\vec{r}$ along the respective axes. Sometimes x, y and z are also called rectangular components.

The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is readily determined by applying the Pythagoras theorem twice. We note that in the right-angled triangle OQP_1,

$$|\overrightarrow{OP_1}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QP_1}|^2} = \sqrt{x^2 + y^2}$$

And in the right-angled triangle OP_1P, we have

$$|\overrightarrow{OP}| = \sqrt{|\overrightarrow{OP_1}|^2 + |\overrightarrow{P_1P}|^2} = \sqrt{(x^2 + y^2) + z^2}$$

Hence, the length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Note:

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, respectively, then

i. The sum (or resultant) of vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

ii. The difference between vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_2)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

iii. Vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

$\vec{b} = \lambda \vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let $\vec{a}$ and $\vec{b}$ be any two vectors, and k and m be any scalars. Then

i. $k\vec{a} + m\vec{a} = (k+m)\vec{a}$

ii. $k(m\vec{a}) = (km)\vec{a}$

iii. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

Remarks

i. One may observe that whatever be the value of λ, vector $\lambda\vec{a}$ is always collinear to vector $\vec{a}$. In fact, two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a non-zero scalar λ such that $\vec{b} = \lambda\vec{a}$. If the vectors $\vec{a}$ and $\vec{b}$ are given in the component form, i.e., $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\Leftrightarrow \quad b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

$$\Leftrightarrow \quad b_1 = \lambda a_1,\ b_2 = \lambda a_2,\ b_3 = \lambda a_3$$

$$\Leftrightarrow \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

ii. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then a_1, a_2, a_3 are also called direction ratios of $\vec{a}$.

iii. In case it is given that l, m, n are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector where α, β and γ are the angles which the vector makes with the x-, y- and z-axes, respectively.

ILLUSTRATION 2.9

A unit vector of modulus 2 is equally inclined to x- and y-axes at an angle $\pi/3$. Find the length of projection of the vector on the z-axis.

Sol. Given that the vector is inclined at an angle $\pi/3$ with both x- and y-axes. Then

$$\cos\alpha = \cos\beta = \frac{1}{2}$$

Also we know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Therefore,

$$\cos^2\gamma = \frac{1}{2}$$

or $$\cos\gamma = \pm\frac{1}{\sqrt{2}}$$

Thus, the given vector is

$$2(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}) = 2\left(\frac{\hat{i}}{2} + \frac{\hat{j}}{2} \pm \frac{\hat{k}}{\sqrt{2}}\right) = \hat{i} + \hat{j} \pm \sqrt{2}\hat{k}$$

Hence, the length of projection of vector on the z-axis is $\sqrt{2}$ units.

ILLUSTRATION 2.10

If the projections of vector $\vec{a}$ on x-, y- and z-axes are 2, 1 and 2 units, respectively, find the angle at which vector $\vec{a}$ is inclined to the z-axis.

Sol. Since projections of vector $\vec{a}$ on x-, y- and z-axes are 2, 1 and 2 units, respectively, we have

$$\text{Vector } \vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$|\vec{a}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

Then $\cos\gamma = \frac{2}{3}$ (where γ is the angle of vector $\vec{a}$ with the z-axis), i.e.,

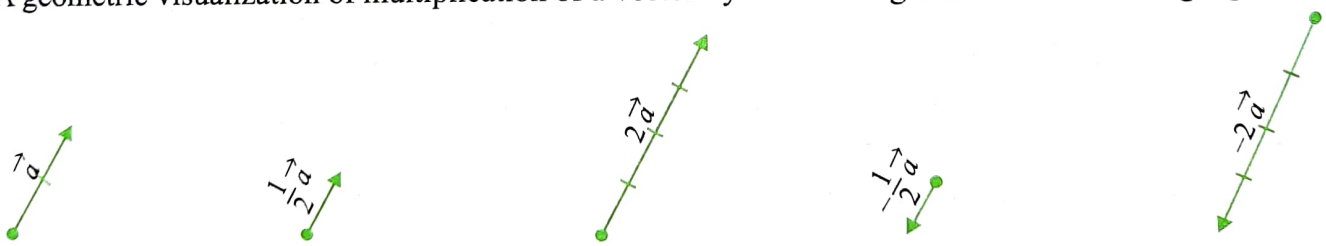
$$\gamma = \cos^{-1}\frac{2}{3}$$

MULTIPLICATION OF A VECTOR BY A SCALAR

Let $\vec{a}$ be a vector and λ a scalar. Then the product of vector $\vec{a}$ by scalar λ, denoted by $\lambda\vec{a}$, is called the multiplication of vector $\vec{a}$ by the scalar λ. Note that $\lambda\vec{a}$ is also a vector, collinear to vector $\vec{a}$. Vector $\lambda\vec{a}$ has the direction same (or opposite) as that of vector $\vec{a}$ if the value of λ is positive (or negative). Also, the magnitude of vector $\lambda\vec{a}$ is $|\lambda|$ times the magnitude of vector $\vec{a}$, i.e.,

$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

A geometric visualization of multiplication of a vector by a scalar is given in the following figure.



When $\lambda = -1$, $\lambda\vec{a} = -\vec{a}$, which is a vector having magnitude equal to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$.

Vector $-\vec{a}$ is called the negative (or additive inverse) of vector $\vec{a}$ and we always have $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

Also, if $\lambda = \dfrac{1}{|\vec{a}|}$, provided $\vec{a} \neq 0$, i.e., $\vec{a}$ is not a null vector, then

$$|\lambda \vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|}|\vec{a}| = 1$$

ILLUSTRATION 2.11

Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.

Sol. Let $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$

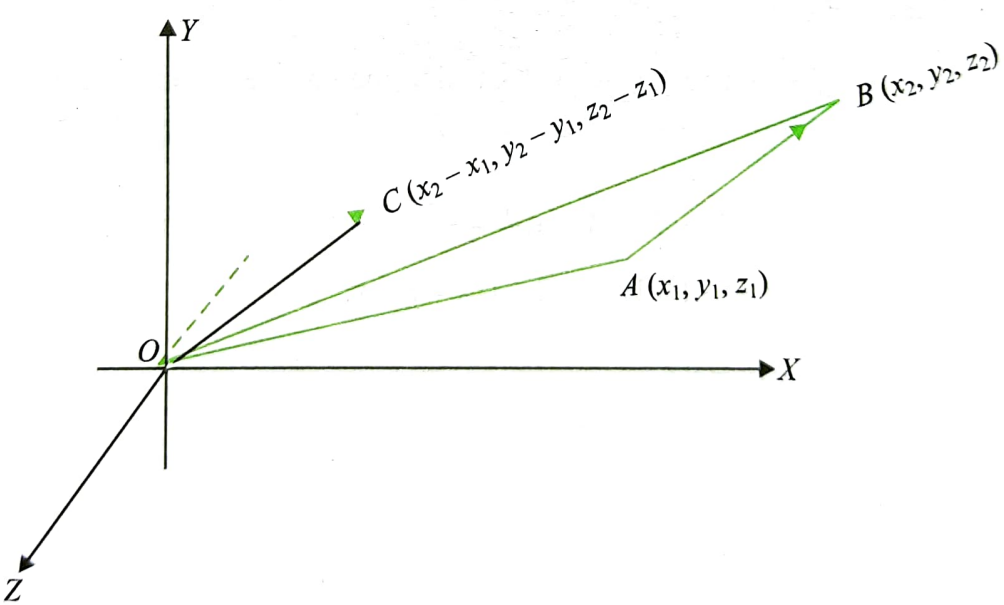
$\therefore$ $|\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25 + 1 + 4} = \sqrt{30}$

$\therefore$ $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}$

Hence, the vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units is given by

$$8\hat{a} = 8\left(\frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}\right) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$$

VECTOR JOINING TWO POINTS



In the figure, vector $\vec{AB}$ is shifted without rotation and placed at origin.

Now vector $\vec{AB} = \vec{OC}$

Since $|\vec{AB}| = |\vec{OC}|$, coordinates of point C are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Hence, Vector $\vec{OC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Thus, $\vec{AB} = \vec{OC} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$= \vec{OB} - \vec{OA}$

= Position vector of B – Position vector of A

Also from above, we have $\vec{OB} = \vec{OA} + \vec{AB}$ which describes triangle rule of vector addition.

Further $\vec{OB} = \vec{OA} + \vec{AB} = \vec{OA} + \vec{OC}$ $(\because \vec{OC} = \vec{AB})$, which describes parallelogram rule of vector addition.

ILLUSTRATION 2.12

Find the unit vector in the direction of vector $\vec{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.

Sol. The given points are P (1, 2, 3) and Q (4, 5, 6). Therefore,

$$\vec{PQ} = (4-1)\,\hat{i} + (5-2)\,\hat{j} + (6-3)\,\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$|\vec{PQ}| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$$

Hence, the unit vector in the direction of $\vec{PQ}$ is

$$\frac{\vec{PQ}}{|\vec{PQ}|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

ILLUSTRATION 2.13

For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.

Sol. The given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$. Therefore,

$$\vec{a} + \vec{b} = (2-1)\,\hat{i} + (-1+1)\,\hat{j} + (2-1)\,\hat{k} = \hat{i} + \hat{k}$$

$$\therefore \quad |\vec{a} + \vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Hence, the unit vector in the direction of $(\vec{a} + \vec{b})$ is

$$\frac{(\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

ILLUSTRATION 2.14

Show that the points A, B and C with position vectors $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right-angled triangle.

Sol. Position vectors of points A, B and C are, respectively given as

$$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k} \text{ and } \vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$$

$\therefore$ $\vec{AB} = \vec{b} - \vec{a} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$

$\vec{BC} = \vec{c} - \vec{b} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\vec{CA} = \vec{a} - \vec{c} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

$\therefore$ $|\vec{AB}|^2 = (-1)^2 + 3^2 + 5^2 = 35$

$|\vec{BC}|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 41$

$|\vec{CA}|^2 = 2^2 + (-1)^2 + 1^2 = 6$

$\therefore$ $|\vec{AB}|^2 + |\vec{CA}|^2 = 36 + 6 = 41 = |\vec{BC}|^2$

Hence, ABC is a right-angled triangle.

ILLUSTRATION 2.15

If $2\overrightarrow{AC} = 3\overrightarrow{CB}$, then prove that $2\overrightarrow{OA} + 3\overrightarrow{OB} = 5\overrightarrow{OC}$ where O is the origin.

Sol. $2\overrightarrow{AC} = 3\overrightarrow{CB}$ or $2(\overrightarrow{OC} - \overrightarrow{OA}) = 3(\overrightarrow{OB} - \overrightarrow{OC})$

or $2\overrightarrow{OA} + 3\overrightarrow{OB} = 5\overrightarrow{OC}$

ILLUSTRATION 2.16

Prove that points $\hat{i} + 2\hat{j} - 3\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $2\hat{i} + 5\hat{j} - \hat{k}$ form a triangle in space.

Sol. Given points are $A(\hat{i} + 2\hat{j} - 3\hat{k}), B(2\hat{i} - \hat{j} + \hat{k}), C(2\hat{i} + 5\hat{j} - \hat{k})$

Vectors $\vec{AB} = \hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{AC} = \hat{i} + 3\hat{j} + 2\hat{k}\,k$

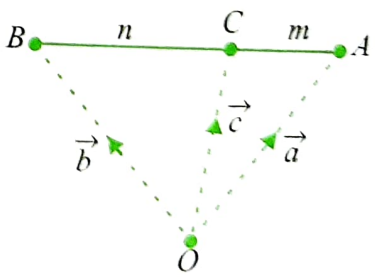
Clearly vectors $\vec{AB}$ and $\vec{AC}$ are non-collinear as there does not exist any real λ for which $\vec{AB} = \lambda\vec{AC}$.

Hence, vectors $\vec{AB}$ and $\vec{AC}$ or the given three points form a triangle.

SECTION FORMULA

Internal Division

Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively and C be a point dividing AB internally in the ratio $m : n$. Then the position vector of C is given by $\overrightarrow{OC} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$.



Proof:

Let O be the origin. Then $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of C which divides AB internally in the ratio $m : n$. Then

$$\frac{AC}{CB} = \frac{m}{n}$$

or $n\overrightarrow{AC} = m\overrightarrow{CB}$

or $n(\text{P.V. of } C - \text{P.V. of } A) = m(\text{P.V. of } B - \text{P.V. of } C)$

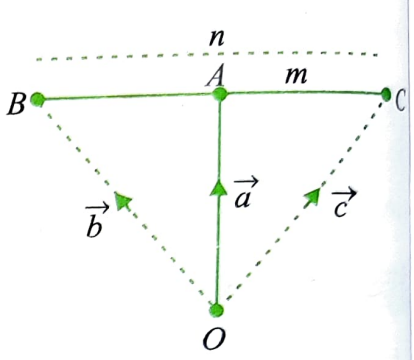
or $n(\vec{c}-\vec{a})=m(\vec{b}-\vec{c})$

or $n\vec{c}-n\vec{a}=m\vec{b}-m\vec{c}$

or $\vec{c}(n+m)=m\vec{b}+n\vec{a}$

or $\vec{c}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$ or $\overrightarrow{OC}=\dfrac{m\vec{b}+n\vec{a}}{m+n}$

External Division



Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively, and C be a point dividing $\overrightarrow{AB}$ externally in the ratio $m : n$. Then the position vector of $\vec{C}$ is given by $\overrightarrow{OC}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$.

Proof:

Let O be the origin. Then $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$. Let $\vec{c}$ be the position vector of point C dividing AB externally in the ratio $m : n$. Then

Then, $\dfrac{AC}{BC}=\dfrac{m}{n}$

or $nAC=mBC$

or $n\overrightarrow{AC}=m\overrightarrow{BC}$

or $n(\text{P.V. of } C-\text{P.V. of } A)=m(\text{P.V. of } C-\text{P.V. of } B)$

or $n\,n(\vec{c}-\vec{a})=m(\vec{c}-\vec{b})$

or $n\vec{c}-n\vec{a}=m\vec{c}-m\vec{b}$

or $\vec{c}(m-n)=m\vec{b}-n\vec{a}$

or $\vec{c}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$ or $\overrightarrow{OC}=\dfrac{m\vec{b}-n\vec{a}}{m-n}$

Notes:

- If C is the midpoint of AB, then it divides AB in the ratio $1 : 1$.

  Therefore, the P.V. of C is $\dfrac{1\cdot\vec{a}+1\cdot\vec{b}}{1+1}=\dfrac{\vec{a}+\vec{b}}{2}$. Thus, the position vector of the midpoint of AB is $\dfrac{1}{2}(\vec{a}+\vec{b})$.

- We have $\vec{c}=\dfrac{m\vec{b}+n\vec{a}}{m+n}=\dfrac{m}{m+n}\vec{b}+\dfrac{n}{m+n}\vec{a}$. Therefore,

  $\vec{c}=\lambda\vec{a}+\mu\vec{b}$, where $\lambda=\dfrac{n}{m+n}$ and $\mu=\dfrac{m}{m+n}$

  Thus, position vector of any point C on $\overrightarrow{AB}$ can always be taken as $\vec{c}=\lambda\vec{a}+\mu\vec{b}$, where $\lambda+m=1$.

- We have $\vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$. Therefore,

$$(m+n)\,\vec{c} = m\vec{b} + n\vec{a}$$

$$n\overrightarrow{OA} + m\overrightarrow{OB} = (m+n)\overrightarrow{OC}$$, where $\vec{C}$ is a point on $\overrightarrow{AB}$ dividing it in the ratio $m : n$.

In ΔABC, having vertices $A(\vec{a}), B(\vec{b})$ and $C(\vec{a})$,

Centroid is $\dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

Incentre is $\dfrac{BC\,\vec{a} + AC\,\vec{b} + AB\,\vec{c}}{AB + AC + AB}$

Orthocentre is $\dfrac{\tan A\,\vec{a} + \tan B\,\vec{b} + \tan C\,\vec{c}}{\tan A + \tan B + \tan C}$

Circumcentre is $\dfrac{\sin 2A\,\vec{a} + \sin 2B\,\vec{b} + \sin 2C\,\vec{c}}{\sin 2A + \sin 2B + \sin 2C}$

ILLUSTRATION 2.17

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$, respectively, in the ratio 2 : 1.

i. Internally **ii.** Externally

Sol. The position vector of point R dividing the line segment joining two points P and Q in the ratio $m : n$ is given by

i. Internally: $\dfrac{m\vec{b} + n\vec{a}}{m+n}$ **ii.** Externally: $\dfrac{m\vec{b} - n\vec{a}}{m-n}$

Position vectors of P and Q are given as

$$\vec{OP} = \hat{i} + 2\hat{j} - \hat{k} \text{ and } \vec{OQ} = -\hat{i} + \hat{j} + \hat{k}$$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2 : 1 is given by

$$\vec{OR} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2+1} = \frac{(-2\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})}{3}$$

$$= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2 : 1 is given by

$$\vec{OR} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2-1} = (-2\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})$$

$$= -3\hat{i} + 3\hat{k}$$

ILLUSTRATION 2.18

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the position vectors of points A, B, C and D, respectively referred to the same origin O such that no three of these points are collinear and $\vec{a} + \vec{c} = \vec{b} + \vec{d}$, then prove that quadrilateral $ABCD$ is a parallelogram.

Sol. Since $\vec{a} + \vec{c} = \vec{b} + \vec{d}$, we have

$$\frac{\vec{a} + \vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2}$$

i.e., Midpoint of AC and BD coincide.

Hence, quadrilateral $ABCD$ is a parallelogram.

ILLUSTRATION 2.19

Find the point of intersection of AB and CD, where $A(6, -7, 0)$, $B(16, -19, -4)$, $C(0, 3, -6)$ and $D(2, -5, 10)$.

Sol. Let AB and CD intersect at P.

Let P divides AB in ratio $\lambda:1$ and CD in ratio $\mu:1$.

Then coordinates of P are $\left(\frac{16\lambda+6}{\lambda+1}, \frac{-19\lambda-7}{\lambda+1}, \frac{-4\lambda}{\lambda+1}\right)$ or $\left(\frac{2\mu}{\mu+1}, \frac{-5\mu+3}{\mu+1}, \frac{10\mu-6}{\mu+1}\right)$

Comparing we have $\lambda = -\frac{1}{3}$ or $\mu = 1$.

Using these values, we get point of intersection as $(1, -1, 2)$.

Here it is also proved that lines AB and CD intersect or points A, B, C and D are coplanar.

ILLUSTRATION 2.20

Find the angle of vector $\vec{a} = 6\hat{i} + 2\hat{j} - 3\hat{k}$ with x-axis.

Sol. $\vec{a} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

or $|\vec{a}| = \sqrt{(6)^2 + (2)^2 + (-3)^2} = 7$

Hence, the angle of vector with the x-axis is $\cos^{-1}\frac{6}{7}$.

ILLUSTRATION 2.21

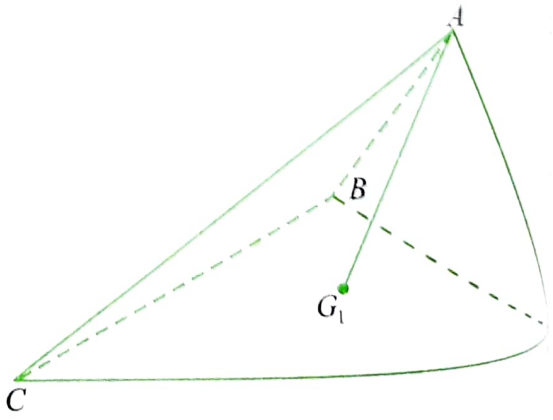
i. Show that the lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.

ii. Show that the joins of the midpoints of the opposite edges of a tetrahedron intersect and bisect each other.

Sol.

i. G_1, the centroid of ΔBCD, is $\frac{\vec{b} + \vec{c} + \vec{d}}{3}$ and A is $\vec{a}$.

The position vector of point G which divides AG_1 in the ratio 3 : 1 is



$$\frac{3\cdot\frac{\vec{b}+\vec{c}+\vec{d}}{3}+1\cdot\vec{a}}{3+1}=\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$$

The symmetry of the result shows that this point will also lie on BG_2, CG_3 and DG_4 (where G_2, G_3, G_4 are centroids of faces ACD, ABD and ABC, respectively). Hence, these four lines are concurrent at point $\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$, which is called the centroid of the tetrahedron.

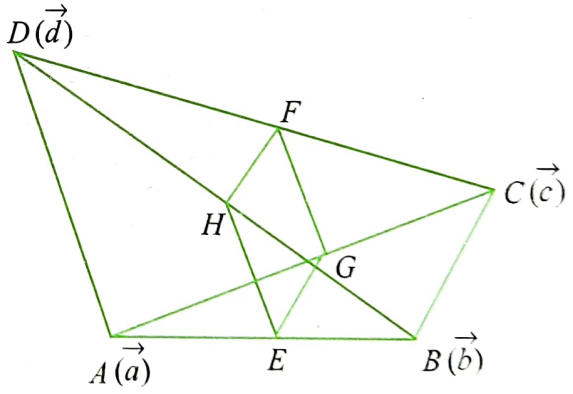
ii. The midpoint of DA is $\frac{\vec{a}+\vec{d}}{2}$ and that of BC is $\frac{\vec{b}+\vec{c}}{2}$ and the midpoint of these midpoints is $\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$ and symmetry of the result proves the fact.

ILLUSTRATION 2.22

The midpoints of two opposite sides of a quadrilateral and the midpoints of the diagonals are the vertices of a parallelogram. Prove this using vectors.

Sol. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of vertices A, B, C and D, respectively.

Let E, F, G and H be the midpoints of AB, CD, AC and BD, respectively.



$$\text{P.V. of } E=\frac{\vec{a}+\vec{b}}{2}$$

$$\text{P.V. of } F=\frac{\vec{c}+\vec{d}}{2}$$

$$\text{P.V. of } G=\frac{\vec{a}+\vec{c}}{2}$$

$$\text{P.V. of } H=\frac{\vec{b}+\vec{d}}{2}$$

$$\overrightarrow{EG}=\text{P.V. of G}-\text{P.V. of } E=\frac{\vec{a}+\vec{c}}{2}-\frac{\vec{a}+\vec{b}}{2}=\frac{\vec{c}-\vec{b}}{2}$$

$$\overrightarrow{HF}=\text{P.V. of F}-\text{P.V. of } H=\frac{\vec{c}+\vec{d}}{2}-\frac{\vec{b}+\vec{d}}{2}=\frac{\vec{c}-\vec{b}}{2}$$

$\therefore$ $\overrightarrow{EG}=\overrightarrow{HF}\Rightarrow EG\parallel HF$ and $EG=HF$

Hence, $EGHF$ is a parallelogram.

ILLUSTRATION 2.23

Check whether the three vectors $2\hat{i}+2\hat{j}+3\hat{k}, -3\hat{i}+3\hat{j}+2\hat{k}$ and $3\hat{i}+4\hat{k}$ form a triangle or not.

Sol. If vectors $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-3\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{c}=3\hat{i}+4\hat{k}$ form a triangle, then we must have $\vec{a}+\vec{b}+\vec{c}=0$.

But for given vectors, $\vec{a}+\vec{b}+\vec{c}\neq 0$. Hence, these vectors do not form a triangle.

ILLUSTRATION 2.24

Find the resultant of vectors $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-4\hat{k}$. Find the unit vector in the direction of the resultant vector.

Sol. Let the resultant vector of $\vec{a}$ and $\vec{b}$ is $\vec{a}+\vec{b}=2\hat{i}+\hat{j}-2\hat{k}=\vec{c}$

Now unit vector in the direction of $\vec{c}$ is $\hat{c}=\dfrac{2\hat{i}+\hat{j}-2\hat{k}}{\sqrt{(2)^2+(1)^2+(-2)^2}}=\dfrac{1}{3}(2\hat{i}+\hat{j}-2\hat{k})$

ILLUSTRATION 2.25

If in parallelogram $ABCD$, diagonal vectors are $\vec{AC}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{BD}=-6\hat{i}+7\hat{j}-2\hat{k}$, then find the adjacent side vectors $\vec{AB}$ and $\vec{AD}$.

Sol. Let $\vec{AB}=\vec{a}$ and $\vec{AD}=\vec{b}$. Then

$$\vec{AC}=\vec{a}+\vec{b} \text{ and } \vec{BD}=\vec{b}-\vec{a}$$

$$\Rightarrow \quad \vec{b}=\frac{\vec{AC}+\vec{BD}}{2} \text{ and } \vec{a}=\frac{\vec{AC}-\vec{BD}}{2}$$

$$\Rightarrow \quad \vec{AB}=-2\hat{i}+35+\hat{k} \text{ and } \vec{AD}=4\hat{i}-2\hat{j}+3\hat{k}$$

ILLUSTRATION 2.26

If two sides of a triangle are $\hat{i}+2\hat{j}$ and $\hat{i}+\hat{k}$, then find the length of the third side.

Sol. Given sides of the triangle are $\vec{a}=\hat{i}+2\hat{j}$ and $\vec{b}=\hat{i}+\hat{k}$.

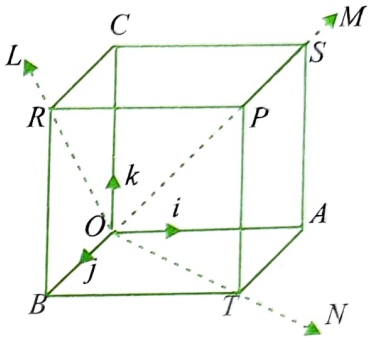
If vector along the third side is $\vec{c}$, then we must have $\vec{a}+\vec{b}+\vec{c}=0$. Then

$$\vec{c}=-(\hat{i}+2\hat{j})-(\hat{i}+\hat{k})=-2\hat{i}-2\hat{j}-\hat{k}$$

Therefore, the length of the third side $|\vec{c}|$ is $\sqrt{(-2)^2+(-2)^2+(-1)^2}=3$.

ILLUSTRATION 2.27

Three coinitial vectors of magnitudes a, $2a$ and $3a$ meet at a point and their directions are along the diagonals of three adjacent faces of a cube. Determine their resultant R. Also prove that the sum of the three vectors determined by the diagonals of three adjacent faces of a cube passing through the same corner, the vectors being directed from the corner, is twice the vector determined by the diagonal of the cube.



Sol. Let the length of an edge of the cube be taken as unity and the vectors represented by OA, OB and OC (let the three coterminous edges of unit be $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively). OR, OS and OT are the three diagonals of the three adjacent faces of the cube along which act the forces of magnitudes a, $2a$ and $3a$, respectively. To find the vectors representing these forces, we will first find unit vectors in these directions and then multiply them by the corresponding given magnitudes of these forces.

Since $\overrightarrow{OR} = \hat{j} + \hat{k}$, the unit vector along OR is $\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$.

Hence, force $\vec{F}_1$ of magnitude a along OR is given by

$$\vec{F}_1 = \frac{a}{\sqrt{2}}(\hat{i}+\hat{k})$$

Similarly, force $\vec{F}_1$ of magnitude $2a$ along OS is $\frac{2a}{\sqrt{2}}(\hat{k}+\hat{i})$ and force $\vec{F}_3$ of magnitude $3a$ along OT is $\frac{3a}{\sqrt{2}}(\hat{i}+\hat{j})$.

If $\vec{R}$ is their resultant, then

$$\begin{aligned}\vec{R} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= \frac{a}{\sqrt{2}}(\hat{j}+\hat{k}) + \frac{2a}{\sqrt{2}}(\hat{k}+\hat{i}) + \frac{3a}{\sqrt{2}}(\hat{i}+\hat{j}) \\ &= \frac{5a}{\sqrt{2}}\hat{i} + \frac{4a}{\sqrt{2}}\hat{j} + \frac{3a}{\sqrt{2}}\hat{k}\end{aligned}$$

Again, $\overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} = \hat{j}+\hat{k}+\hat{i}+\hat{k}+\hat{i}+\hat{j}$

$$= 2(\hat{i}+\hat{j}+\hat{k})$$

Also $\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{TP} = (\hat{i}+\hat{j}+\hat{k})$ $\quad (\because \overrightarrow{OT} = \hat{i}+\hat{j}$ and $\overrightarrow{TP} = \overrightarrow{OC} = \hat{k})$

$$\overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} = 2\overrightarrow{OP}$$

ILLUSTRATION 2.28

The axes of coordinates are rotated about the z-axis through an angle of $\pi/4$ in the anticlockwise direction and the components of a vector are $2\sqrt{2}$, $3\sqrt{2}$, 4. Prove that the components of the same vector in the original system are -1, 5, 4.

Sol. If $\hat{i}, \hat{j}, \hat{k}$ are the new unit vectors along the coordinate axes, then

$$\vec{a} = 2\sqrt{2}\,\hat{i} + 2\sqrt{2}\,\hat{j} + 4\hat{k}$$

$\hat{i}, \hat{j}, \hat{k}$ are obtained by rotating by 45° about the z-axis.

Then $\hat{i}$ is replaced by $\hat{i}\cos 45° + \hat{j}\sin 45° = \dfrac{\hat{i}+\hat{j}}{\sqrt{2}}$;

$\hat{j}$ is replaced by $-\hat{i}\cos 45° + \hat{j}\sin 45° = \dfrac{-\hat{i}+\hat{j}}{\sqrt{2}}$;

and $\hat{k} = \hat{k}$,

$$\therefore \quad \vec{a} = 2\sqrt{2}\left[\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right] + 3\sqrt{2}\left[\frac{-\hat{i}+\hat{j}}{\sqrt{2}}\right] + 4\hat{k}$$

$$= (2-3)\,\hat{i} + (2+3)\,\hat{j} + 4\hat{k}$$

$$= -\hat{i} + 5\hat{j} + 4\hat{k}$$

ILLUSTRATION 2.29

If the resultant of two forces is equal in magnitude to one of the components and perpendicular to it in direction, find the other component using the vector method.

Sol. Let P be horizontal in the direction of unit vector $\hat{i}$. The resultant is also P but perpendicular to it in the direction of unit vector $\hat{j}$. If Q is the other force making an angle θ (obtuse) as the resultant is perpendicular to P, then the two forces are $P\hat{i}$ and $Q\cos\theta\,\hat{i} + Q\sin\theta\,\hat{j}$. Their resultant is $P\hat{j}$. Therefore,

$$P\hat{j} = P\hat{i} + (Q\cos\theta\,\hat{i} + Q\sin\theta\,\hat{j})$$

Comparing the coefficients of $\hat{i}$ and $\hat{j}$, we get

$$P + Q\cos\theta = 0 \quad \text{and} \quad Q\sin\theta = P$$

or $$Q\cos\theta = -P \text{ and } Q\sin\theta = P$$

Squaring and adding $Q = P\sqrt{2}$ and dividing give

$$\tan\theta = -1$$

$$\theta = 135°$$

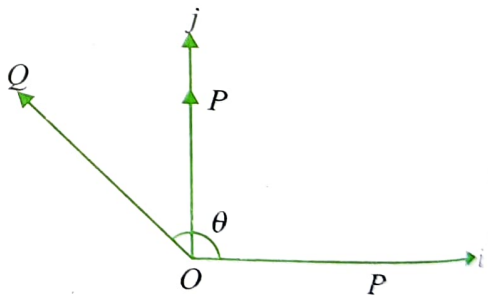


ILLUSTRATION 2.30

A man travelling towards east at 8 km/h finds that the wind seems to blow directly from the north. On doubling the speed, he finds that it appears to come from the north-east. Find the velocity of the wind.

Sol. Velocity of wind relative to man

= Actual velocity of wind – Actual velocity of man

Let $\hat{i}$ and $\hat{j}$ represent unit vectors along east and north. Let the actual velocity of wind be given by $x\hat{i} + y\hat{j}$.

In the first case, the man's velocity is $8\hat{i}$ and that of the wind blowing from the north relative to the man is $-p\hat{j}$. Therefore,

$$-p\hat{j} = (x\hat{i} + y\hat{j}) - 8\hat{i} \qquad \text{[from Eq. (i)]}$$

Comparing coefficients, $x - 8 = 0, y = -p$ (ii)

In the second case, when the man doubles his speed, wind seems to come from the north-east direction, i.e.,

$$-q(\hat{i} + \hat{j}) = (x\hat{i} + y\hat{j}) - 16\hat{i}$$

$\therefore$ $x - 16 = -q, y = -q$ (iii)

Putting $x = 8$, we get $q = 8$

$$y = -8$$

Hence, the velocity of wind is $x\hat{i} + y\hat{j} = 8(\hat{i} - \hat{j})$

Its magnitude is $\sqrt{(8^2 + 8^2)} = 8\sqrt{2}$ and $\tan\theta = -1$ or $\theta = -45°$

Hence, its direction is from the north to the west.

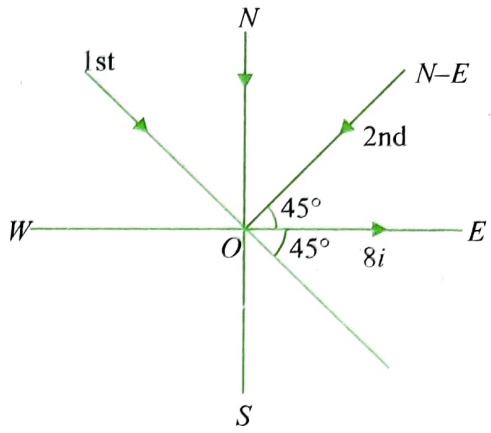


ILLUSTRATION 2.31

OABCDE is a regular hexagon of side 2 units in the *XY*-plane in the first quadrant. *O* being the origin and *OA* taken along the *x*-axis. A point *P* is taken on a line parallel to the *z*–axis through the centre of the hexagon at a distance of 3 units from *O* in the positive *Z* direction. Then find vector $\overrightarrow{AP}$.

Sol. $G \equiv (\hat{i} + \sqrt{3}\,\hat{j})$

Let position vector of *P* be $\vec{p}$

$\because$ $\overrightarrow{GP} \parallel \hat{k}$

Then $\vec{p} - (\hat{i} + \sqrt{3}\,\hat{j}) = \lambda\hat{k}$

$\therefore$ $\vec{p} = \hat{i} + \sqrt{3}\,\hat{j} + \lambda\hat{k}$

Also $\left|\overrightarrow{OP}\right| = 3$

$\Rightarrow$ $\sqrt{1 + 3 + \lambda^2} = 3$

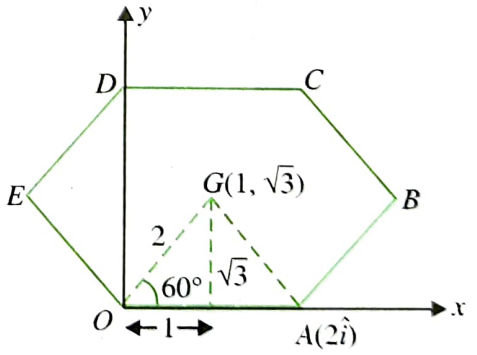
or $\lambda^2 = 5$

or $\lambda = \pm\sqrt{5}$

$\Rightarrow$ $\vec{p} = \hat{i} + \sqrt{3}\,\hat{j} \pm \sqrt{5}\,\hat{k}$

For positive *z*-axis, $\vec{p} = \hat{i} + \sqrt{3}\,\hat{j} + \sqrt{5}\,\hat{k}$

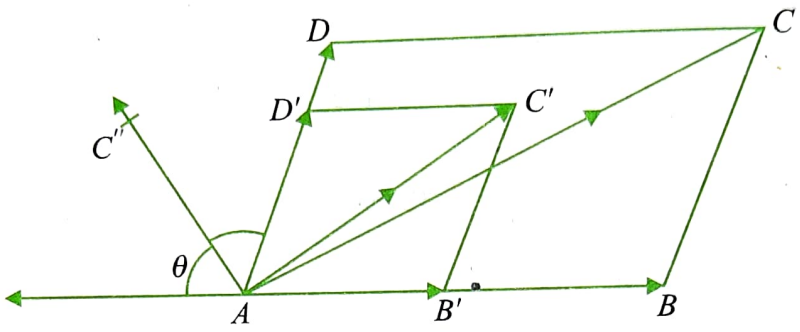
So $\overrightarrow{AP} = \vec{p} - 2\hat{i} = -\hat{i} + \sqrt{3}\,\hat{j} + \sqrt{5}\,\hat{k}$



VECTOR ALONG THE BISECTOR OF GIVEN TWO VECTORS

We know that the diagonal in a parallelogram is not necessarily the bisector of the angle formed by two adjacent sides. However, the diagonal in a rhombus bisects the angle between two adjacent sides.

Consider vectors $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$ forming a parallelogram $ABCD$ as shown in the figure.



Consider the two unit vectors along the given vectors, which form a rhombus $AB'C'D'$. Now

$$\overrightarrow{AB'} = \frac{\vec{a}}{|\vec{a}|} \text{ and } \overrightarrow{AD'} = \frac{\vec{b}}{|b|}$$

$$\therefore \quad \overrightarrow{AC'} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$$

So any vector along the bisector is $\lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right)$.

Similarly, any vector along the external bisector is $\overrightarrow{AC''} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} - \frac{\vec{b}}{|\vec{b}|}\right)$.

ILLUSTRATION 2.32

If $\vec{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\vec{b} = -2\hat{i} - \hat{j} + 2\hat{k}$, determine vector $\vec{c}$ along the internal bisector of the angle between vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{c}| = 5\sqrt{6}$.

Sol. $\hat{a} = \frac{1}{9}(7\hat{i} - 4\hat{j} - 4\hat{k})$

$$\hat{b} = \frac{1}{3}(-2\hat{i} - \hat{j} + 2\hat{k})$$

$$\vec{c} = \lambda[\hat{a} + \hat{b}] = \lambda\frac{1}{9}(\hat{i} - 7\hat{j} + 2\hat{k})$$

$$|\vec{c}| = 5\sqrt{6}$$

$$\Rightarrow \quad \frac{\lambda^2}{81}(1 + 49 + 4) = 25 \times 6$$

$$\lambda^2 = \frac{25 \times 6 \times 81}{54} = 225$$

$$\lambda = \pm 15$$

Putting the value of λ in (i), we get

$$\vec{c} = \pm \frac{5}{3}(\hat{i} - 7\hat{j} + 2\hat{k})$$

ILLUSTRATION 2.33

Find a unit vector $\vec{c}$ if $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between vectors $\vec{c}$ and $3\hat{i} + 4\hat{j}$.

Sol. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$, where $x^2 + y^2 + z^2 = 1$. (i)

Unit vector along $3\hat{i} + 4\hat{j}$ is $\frac{3\hat{i} + 4\hat{j}}{5}$.

The bisector of these two is $-\hat{i} + \hat{j} - \hat{k}$ (given). Therefore,

$$-\hat{i} + \hat{j} - \hat{k} = \lambda\left(x\hat{i} + y\hat{j} + z\hat{k} + \frac{3\hat{i} + 4\hat{j}}{5}\right)$$

$$-\hat{i} + \hat{j} - \hat{k} = \frac{1}{5}\lambda[(5x + 3)\hat{i} + (5y + 4)\hat{j} + 5z\hat{k}] \quad \text{(ii)}$$

$$\frac{\lambda}{5}(5x + 3) = -1, \frac{\lambda}{5}(5y + 4) = 1, \frac{\lambda}{5}5z = -1$$

$$x = -\frac{5 + 3\lambda}{5\lambda}, y = \frac{5 - 4\lambda}{5\lambda}, z = -\frac{1}{\lambda}$$

Putting these values in (i), i.e., $x^2 + y^2 + z^2 = 1$, we get

$$(5 + 3\lambda)^2 + (5 - 4\lambda)^2 + 25 = 25\lambda^2$$

$$25\lambda^2 - 10\lambda + 75 = 25\lambda^2$$

$$\lambda = \frac{15}{2}$$

$$\therefore \quad \vec{c} = \frac{1}{15}(-11\hat{i} + 10\hat{j} - 2\hat{k})$$

CONCEPT APPLICATION EXERCISE 2.1

1. Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$..
2. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.
3. Find the direction cosines of the vector joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$ directed from A to B.
4. The position vectors of P and Q are $5\hat{i} + 4\hat{j} + a\hat{k}$ and $-\hat{i} + 2\hat{j} - 2\hat{k}$, respectively. If the distance between them is 7, then find the value of a.

5. Given three points are $A(-3, -2, 0)$, $B(3, -3, 1)$ and $C(5,0,2)$. Then find a vector having the same direction as that of $\overrightarrow{AB}$ and magnitude equal to $|\overrightarrow{AC}|$

6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

7. Show that the points $A(1, -2, -8)$, $B(5, 0, -2)$ and $C(11, 3, 7)$ are collinear, and find the ratio in which B divides AC.

8. If $ABCD$ is a rhombus whose diagonals cut at the origin O, then prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{O}$.

9. Let D, E and F be the middle points of the sides BC, CA and AB, respectively of a triangle ABC. Then prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$.

10. Let $ABCD$ be a parallelogram whose diagonals intersect at P and let O be the origin. Then prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OP}$.

11. If A, B, C and D are any four points and E and F are the middle points of AC and BD, respectively then prove that $\overrightarrow{CB} + \overrightarrow{CD} + \overrightarrow{AD} + \overrightarrow{AB} = 4\overrightarrow{EF}$.

12. If $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$, then A, B and C are (where O is the origin)

a. coplanar **b.** collinear **c.** non-collinear **d.** none of these

13. If the sides of an angle are given by vectors $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, then find the internal bisector of the angle.

14. $ABCD$ is a parallelogram. If L and M be the middle points of BC and CD, respectively express $\overrightarrow{AL}$ and $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Also show that $\overrightarrow{AL} + \overrightarrow{AM} = (3/2)\overrightarrow{AC}$.

15. $ABCD$ is a quadrilateral and E the point of intersection of the lines joining the middle points of opposite sides. Show that the resultant of $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ is equal to $4\overrightarrow{OE}$, where O is any point.

16. What is the unit vector parallel to $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$? What vector should be added to $\vec{a}$ so that the resultant is the unit vector $\hat{i}$?

17. The position vectors of points A and B w.r.t. the origin are $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 2\hat{k}$, respectively. Determine vector $\overrightarrow{OP}$ which bisects angle AOB, where P is a point on AB.

18. If $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are the position vectors of three collinear points and scalars p and q exist such that $\vec{r}_3 = p\vec{r}_1 + q\vec{r}_2$, then show that $p + q = 1$.

19. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 1 inclined at 120°, then find the angle between $\vec{b}$ and $\vec{b} - \vec{a}$.

20. Find the vector of magnitude 3, bisecting the angle between the vectors $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

ANSWERS

1. $\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$
2. $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$
3. $\left(-\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)$
4. $a=-5, 1$
5. $\frac{6}{\sqrt{19}}(6\hat{i}-\hat{j}+\hat{k})$
6. $\pm\frac{3\sqrt{10}\,\hat{i}}{2}\pm\frac{\sqrt{10}}{2}\hat{j}$
7. 2 : 3
12. (b)
13. $\frac{1}{3}(3\hat{i}-\hat{j}+4\hat{k})$
14. $\overrightarrow{AL}=\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AD}, \overrightarrow{AM}=\overrightarrow{AD}+\frac{1}{2}\overrightarrow{AB}$
16. $\frac{1}{\sqrt{29}}(3\hat{i}+4\hat{j}-2\hat{k}); -2\hat{i}-4\hat{j}+2\hat{k}$
17. $2(\hat{i}+\hat{j}-\hat{k})$
19. 30°
20. $\frac{3(3\hat{i}-\hat{j})}{\sqrt{10}}$ or $\frac{3(\hat{i}+3\hat{j}-2\hat{k})}{\sqrt{14}}$

LINEAR COMBINATION OF VECTORS

LINEAR COMBINATION OF VECTORS

A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$, if there exist scalars $m_1, m_2, \cdots, m_n$ such that $\vec{r}=m_1\vec{a}_1+m_2\vec{a}_2+\cdots+m_n\vec{a}_n$.

LINEARLY INDEPENDENT VECTORS

A system of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly independent if

$$m_1\vec{a}_1+m_2\vec{a}_2+\cdots+m_n\vec{a}_n=\vec{0} \Rightarrow m_1=m_2=\cdots=m_n=0$$

It can be easily verified that

i. A pair of non-collinear vectors is linearly independent.

Proof:

Let $\vec{a}_1$ and $\vec{a}_2$ are non-collinear vectors such that $m_1\vec{a}_1+m_2\vec{a}_2=\vec{O}$

Let $m_1, m_2 \neq 0$

$$\Rightarrow \quad \vec{a}_1=-\frac{m_2}{m_1}\vec{a}_2$$

This means $\vec{a}_1$ and $\vec{a}_2$ are collinear, which contradicts the given fact.

Hence, $m_1, m_2 = 0$

ii. A triad of non-coplanar vector is linearly independent.

LINEARLY DEPENDENT VECTORS

A set of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly dependent if there exist scalars $m_1, m_2, \cdots, m_n$, not all zero, such that $m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_n\vec{a}_n = \vec{0}$.

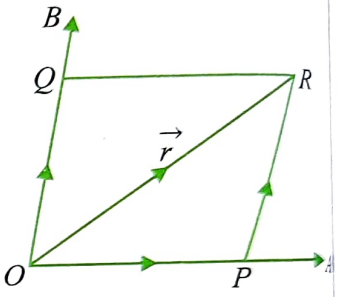
It can be easily verified that

i. A pair of collinear vectors is linearly dependent.

ii. A triad of coplanar vectors is linearly dependent.

Theorem 1

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, then every vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be expressed in one and only one way as a linear combination $x\vec{a} + y\vec{b}$; x and y being scalars.



Proof:

i. Let O be any point such that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

As $\vec{r}$ is coplanar with $\vec{a}$ and $\vec{b}$, the lines OA, OB and OR are coplanar. Through R, draw lines parallel to OA and OB, meeting them at P and Q, respectively. Clearly,

$$\overrightarrow{OP} = x\,\overrightarrow{OA} = x\vec{a} \quad (\because \overrightarrow{OP} \text{ and } \overrightarrow{OA} \text{ are collinear vectors})$$

Also $$\overrightarrow{OQ} = y\,\overrightarrow{OB} = y\vec{b} \quad (\because \overrightarrow{OQ} \text{ and } \overrightarrow{OB} \text{ are collinear vectors})$$

$$\vec{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + \overrightarrow{OQ} \quad (\because \overrightarrow{OQ} \text{ and } \overrightarrow{PR} \text{ are equal})$$

$$= x\vec{a} + y\vec{b} \quad (i)$$

Thus, $\vec{r}$ can be expressed in one way as a linear combination $x\vec{a} + y\vec{b}$.

ii. To prove that this resolution is unique, let $\vec{r} = x'\vec{a} + y'\vec{b}$ be another representation of $\vec{r}$ as a linear combination of $\vec{a}$ and $\vec{b}$. Then,

$$x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b}$$

or $$(x - x')\vec{a} + (y - y')\vec{b} = \vec{0}$$

Since $\vec{a}$ and $\vec{b}$ are non-collinear vectors, we must have

$$x - x' = 0,\ y - y' = 0$$

i.e., $$x = x',\ y = y'$$

Thus, the representation is unique.

Note:

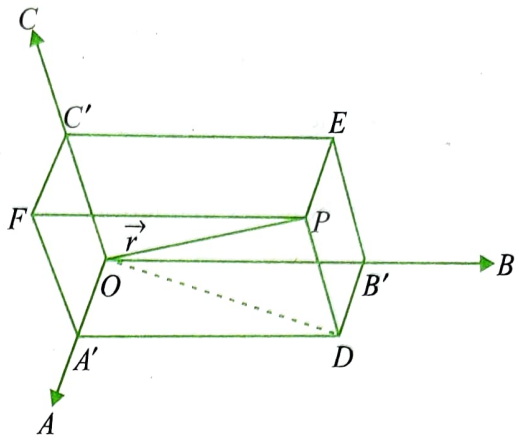
If OA and OB are perpendicular, then these two lines can be taken as the x- and the y-axes, respectively. Let $\hat{i}$ be the unit vector along the x-axis and $\hat{j}$ be the unit vector along the y-axis. Therefore, we have

$$\vec{r} = x\hat{i} + y\hat{j}$$

Also $$r = \sqrt{x^2 + y^2}$$

Theorem 2

If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, then any vector $\vec{r}$ can be uniquely expressed as a linear combination $x\vec{a} + y\vec{b} + z\vec{c}$; x, y and z being scalars.



Proof:

i. Take any point O so that

$$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c} \text{ and } \overrightarrow{OP} = \vec{r}.$$

On OP as diagonal, construct a parallelpiped having edges OA', OB' and OC' along OA, OB and OC, respectively. Then there exist three scalars x, y and z such that

$$\overrightarrow{OA'} = x\,\overrightarrow{OA} = x\vec{a}, \overrightarrow{OB'} = y\,\overrightarrow{OB} = y\vec{b}, \overrightarrow{OC'} = z\,\overrightarrow{OC} = z\vec{c}$$

$$\therefore \quad \vec{r} = \overrightarrow{OP}$$

$$= \overrightarrow{OA'} + \overrightarrow{A'P}$$

$$= \overrightarrow{OA'} + \overrightarrow{A'D} + \overrightarrow{DP} \qquad \text{(by definition of addition of vectors)}$$

$$= \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}$$

$$x\vec{a} + y\vec{b} + z\vec{c} \qquad \text{(i)}$$

Thus $\vec{r}$ can be represented as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$.

ii. To prove that this representation is unique, let, if possible, $\vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$ be another representation of $\vec{r}$ as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$. (ii)

Then from (i) and (ii), we have

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$$

or $$(x - x')\vec{a} + (y - y')\vec{b} + (z - z')\vec{c} = \vec{0}$$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are independent, $x - x' = 0$, $y - y' = 0$ and $z - z' = 0$, or $x = x'$, $y = y'$ and $z = z'$. Hence proved.

Theorem 3

If vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are coplanar, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Proof:

If vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then there exist scalars λ and μ such that $\vec{c} = \lambda \vec{a} + \mu \vec{b}$. Hence,

$$c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k} = \lambda (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) + \mu (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

Now $\hat{i}, \hat{j}$ and $\hat{k}$ are non-coplanar and hence independent. Then,

$$c_1 = \lambda a_1 + \mu b_1,\ c_2 = \lambda a_2 + \mu b_2 \text{ and } c_3 = \lambda a_3 + \mu b_3$$

The above system of equations in terms of λ and μ is consistent. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Similarly, if vectors $x_1 \vec{a} + y_1 \vec{b} + z_1 \vec{c}$, $x_2 \vec{a} + y_2 \vec{b} + z_2 \vec{c}$ and $x_3 \vec{a} + y_3 \vec{b} + z_3 \vec{c}$ are coplanar (where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar). Then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$ can be proved with the same arguments.

To prove that four points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ and $D(\vec{d})$ are coplanar, it is just sufficient to prove that vectors $\overrightarrow{AB}, \overrightarrow{BD}$ and $\overrightarrow{CD}$ are coplanar.

Notes:

- Two collinear vectors are always linearly dependent.
- Two non-collinear non-zero vectors are always linearly independent.
- Three coplanar vectors are always linearly dependent.
- Three non-coplanar non-zero vectors are always linearly independent.
- More than three vectors are always linearly dependent.
- Three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear if and only if there exist scalars x, y and z not all zero such that **(i)** $x\vec{a} + y\vec{b} + z\vec{c} = 0$ and **(ii)** $x + y + z = 0$.

  Proof:

  Let us suppose that points A, B and C are collinear and their position vectors are $\vec{a}, \vec{b}$ and $\vec{c}$, respectively. Let C divide the join of $\vec{a}$ and $\vec{b}$ in the ratio $y : x$. Then,

  $$\vec{c} = \frac{x\vec{a} + y\vec{b}}{x + y}$$

  or $$x\vec{a} + y\vec{b} - (x + y)\vec{c} = \vec{0}$$

  or $$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}, \text{ where } z = -(x + y)$$

  Also, $$x + y + z = x + y - (x + y) = 0.$$

  Conversely, let $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$. Therefore,

  $$x\vec{a} + y\vec{b} = -z\vec{c} = (x + y)\vec{c} \quad (\because x + y = -z)$$

or $\vec{c} = \dfrac{x\vec{a} + y\vec{b}}{x+y}$

This relation shows that $\vec{c}$ divides the join of $\vec{a}$ and $\vec{b}$ in the ratio $y : x$. Hence, the three points A, B and C are collinear.

- Four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if there exist scalars x, y, z and w (sum of any two is not zero) such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$ with $x + y + z + w = 0$.

Proof:

$$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$$

or $$x\vec{a} + y\vec{b} = -(z\vec{c} + w\vec{d}) \quad \text{(i)}$$

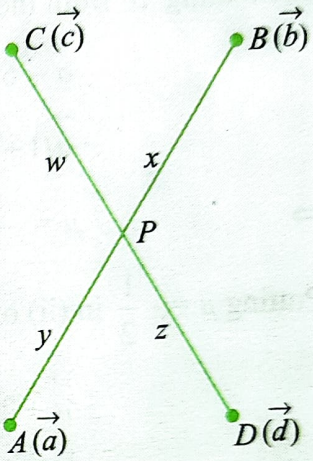
$$x + y + z + w = 0$$

or $$x + y = -(w + z) \quad \text{(ii)}$$

From (i) and (ii), we have $\dfrac{x\vec{a} + y\vec{b}}{x+y} = \dfrac{z\vec{c} + w\vec{d}}{z+w}$

Thus, there is point P

$\Rightarrow$ $\dfrac{x\vec{a} + y\vec{b}}{x+y} = \dfrac{z\vec{c} + w\vec{d}}{z+w}$ (iii)



$\dfrac{x\vec{a} + y\vec{b}}{x+y}$ is the position vector of a point on AB which divides it in the ratio $y : x$.

$\dfrac{z\vec{c} + w\vec{d}}{z+w}$ is the position vector of a point on CD which divides it in the ratio $w : z$.

From (iii), these points are coincident; hence, the points are coplanar.

ILLUSTRATION 2.34

The vectors $2\hat{i} + 3\hat{j}, 5\hat{i} + 6\hat{j}$ and $8\hat{i} + \lambda\hat{j}$ have their initial points at (1, 1). Find the value of λ so that the vectors terminate on one straight line.

Sol. Since the vectors $2\hat{i} + 3\hat{j}$ and $5\hat{i} + 6\hat{j}$ have (1, 1) as the initial point, their terminal points are (3, 4) and (6, 7), respectively. The equation of the line joining these two points is $x - y + 1 = 0$. The terminal point of $8\hat{i} + \lambda\hat{j}$ is $(9, \lambda + 1)$. Since the vectors terminate on the same straight line, $(9, \lambda + 1)$ lies on $x - y + 1 = 0$.

Therefore,

$$9 - \lambda - 1 + 1 = 0$$

or $$\lambda = 9$$

ILLUSTRATION 2.35

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-zero vectors, no two of which are collinear, $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$, then find the value of $|\vec{a}+2\vec{b}+6\vec{c}|$.

Sol. Given $\vec{a}+2\vec{b}=\lambda\vec{c}$ (i)

and $\vec{b}+3\vec{c}=\mu\vec{a}$, (ii)

where no two of $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear vectors.

Eliminating $\vec{b}$ from the above relations, we have

$$\vec{a}-6\vec{c}=\lambda\vec{c}-2\mu\vec{a}$$

$$\vec{a}(1+2\mu)=(\lambda+6)\vec{c}$$

$\Rightarrow$ $\mu=-\frac{1}{2}$ and $\lambda=-6$ as $\vec{a}$ and $\vec{c}$ are non-collinear.

Putting $\mu=-\frac{1}{2}$ in (ii) or $\lambda=-6$ in (i), we get

$$\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$$

or $$\left|\vec{a}+2\vec{b}+3\vec{c}\right|=0$$

ILLUSTRATION 2.36

i. Prove that the points $\vec{a}-2\vec{b}+3\vec{c}$, $2\vec{a}+3\vec{b}-4\vec{c}$ and $-7\vec{b}+10\vec{c}$ are collinear, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.

ii. Prove that the points $A(1, 2, 3)$, $B(3, 4, 7)$ and $C(-3, -2, -5)$ are collinear. Find the ratio in which point C divides AB.

Sol.

i. Let the given points be A, B and C. Therefore,

$$\overrightarrow{AB} = \text{P.V. of } B - \text{P.V. of } A$$
$$= (2\vec{a}+3\vec{b}-4\vec{c})-(\vec{a}-2\vec{b}+3\vec{c})$$
$$= \vec{a}+5\vec{b}-7\vec{c}$$

$$\overrightarrow{AC} = \text{P.V. of } C - \text{P.V. of } A$$
$$= (-7\vec{b}+10\vec{c})-(\vec{a}-2\vec{b}+3\vec{c})$$
$$= -\vec{a}-5\vec{b}+7\vec{c} = -\overrightarrow{AB}$$

Since $\overrightarrow{AC} = -\overrightarrow{AB}$, it follows that the points A, B and C are collinear.

ii. Let C divide AB in the ratio $k:1$; then $C(-3,-2,-5) \equiv \left(\frac{3k+1}{k+1}, \frac{4k+2}{k+1}, \frac{7k+3}{k+1}\right)$

$$\Rightarrow \quad \frac{3k+1}{k+1} = -3, \frac{4k+2}{k+1} = -2 \text{ and } \frac{7k+3}{k+1} = -5$$

$$\Rightarrow \quad k = -\frac{2}{3} \text{ from all relations}$$

Hence, C divides AB externally in the ratio 2:3.

ILLUSTRATION 2.37

Check whether the given three vectors are coplanar or non-coplanar:

$-2\hat{i}-2\hat{j}+4\hat{k}, -2\hat{i}+4\hat{j}-2\hat{k}, 4\hat{i}-2\hat{j}-2\hat{k}$

Sol. Given vectors are $-2\hat{i}-2\hat{j}+4\hat{k}, -2\hat{i}+4\hat{j}-2\hat{k}, 4\hat{i}-2\hat{j}-2\hat{k}$

$$\Rightarrow \quad \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = 16+16+16-64+8+8 = 0$$

Hence, the vectors are coplanar.

ILLUSTRATION 2.38

Prove that the four points $6\hat{i}-7\hat{j}, 16\hat{i}-19\hat{j}-4\hat{k}, 3\hat{j}-6\hat{k}$ and $2\hat{i}+5\hat{j}+10\hat{k}$ form a tetrahedron in space.

Sol. Given points are $A(6\hat{i}-7\hat{j}), B(16\hat{i}-19\hat{j}-4\hat{k}), C(3\hat{j}-6\hat{k}), D(2\hat{i}+5\hat{j}+10\hat{k})$

Hence, vectors $\vec{AB} = 10\hat{i}-12\hat{j}-4\hat{k}$, $\vec{AC} = -6\hat{i}+10\hat{j}-6\hat{k}$ and $\vec{AD} = -4\hat{i}+12\hat{j}+10\hat{k}$

Now determinant of coefficients of $\vec{AB}, \vec{AC}, \vec{AD}$ is

$$\begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix} = 10(100+72)+12(-60-24)-4(-72+40) \neq 0$$

Hence, the given points are non-coplanar and therefore form a tetrahedron in space.

ILLUSTRATION 2.39

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, show that points $l_1\vec{a}+m_1\vec{b}, l_2\vec{a}+m_2\vec{b}$ and $l_3\vec{a}+m_3\vec{b}$ are collinear if $\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$

Sol. We know that three points having P.V.s $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear if there exists a relation of the form $x\vec{a}+y\vec{b}+z\vec{c} = \vec{0}$, where $x+y+z=0$.

Now $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ gives

$$x(l_1\vec{a} + m_1\vec{b}) + y(l_2\vec{a} + m_2\vec{b}) + z(l_3\vec{a} + m_3\vec{b}) = \vec{0}$$

or $$(xl_1 + yl_2 + zl_3)\vec{a} + (xm_1 + ym_2 + zm_3)\vec{b} = \vec{0}$$

Since $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, it follows that

$$xl_1 + yl_2 + zl_3 = 0 \quad \text{(i)}$$

$$xm_1 + ym_2 + zm_3 = 0 \quad \text{(ii)}$$

Because otherwise one is expressible as a scalar multiple of the other which would mean that $\vec{a}$ and $\vec{b}$ are collinear. Also

$$x + y + z = 0 \quad \text{(iii)}$$

Eliminating x, y and z from (i), (ii) and (iii), we get

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Alternate method:

$A(l_1\vec{a} + m_1\vec{b})$, $B(l_2\vec{a} + m_2\vec{b})$ and $C(l_3\vec{a} + m_3\vec{b})$ are collinear.

$\Rightarrow$ Vectors $= (l_2 - l_3)\vec{a} + (m_2 - m_3)\vec{b}$ and $\overrightarrow{AB} = (l_1 - l_2)\vec{a} + (m_1 - m_2)\vec{b}$ are collinear.

$\Rightarrow$ $$\frac{l_1 - l_2}{l_2 - l_3} = \frac{m_1 - m_2}{m_2 - m_3}$$

$\Rightarrow$ $$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

ILLUSTRATION 2.40

Vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find for what value of x vectors $\vec{c} = (x-2)\vec{a} + \vec{b}$ and $\vec{d} = (2x+1)\vec{a} - \vec{b}$ are collinear?

Sol. Both the vectors $\vec{c}$ and $\vec{d}$ are non-zero as the coefficients of $\vec{b}$ in both are non-zero.

Two vectors $\vec{c}$ and $\vec{d}$ are collinear if one of them is a linear multiple of the other. Therefore,

$$\vec{d} = \lambda\vec{c}$$

or $$(2x+1)\vec{a} - \vec{b} = \lambda\{(x-2)\vec{a} + \vec{b}\} \quad \text{(i)}$$

or $$\{(2x+1) - \lambda(x-2)\}\vec{a} - (1+\lambda)\vec{b} = 0$$

The above expression is of the form $p\vec{a} + q\vec{b} = 0$, where $\vec{a}$ and $\vec{b}$ are non-collinear, and hence we have $p = 0$ and $q = 0$. Therefore,

$$2x + 1 - \lambda(x - 2) = 0 \quad \text{(ii)}$$

and
$$1 + \lambda = 0 \quad \text{(iii)}$$

From (iii), $\lambda = -1$, and putting this value in (i), we get $x = \frac{1}{3}$

Alternate method:

$\vec{c} = (x-2)\vec{a} + \vec{b}$ and $\vec{d} = (2x+1)\vec{a} - \vec{b}$ are collinear.

If $\frac{x-2}{2x+1} = \frac{1}{-1}$, then $x = \frac{1}{3}$

ILLUSTRATION 2.41

The median AD of the triangle ABC is bisected at E and BE meets AC at F. Find $AF : FC$.

Sol. Taking A at the origin

Let P.V. of B and C be $\vec{b}$ and $\vec{c}$, respectively.

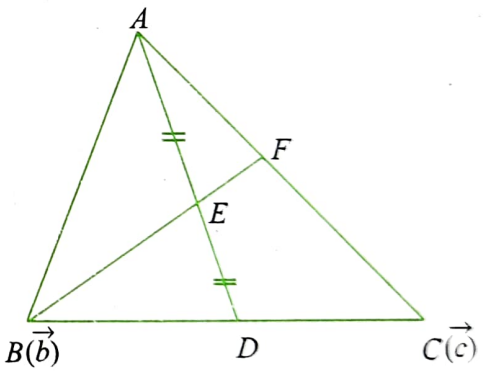
P.V. of D is $\frac{\vec{b} + \vec{c}}{2}$ and P.V. of E is $\frac{\vec{b} + \vec{c}}{4}$

Let $AF:FC = p:1$.

Then position vector of F is $\frac{p\vec{c}}{p+1}$. (i)

Let $BF:EF = q : 1$.

The position vector of F is $\frac{q\frac{(\vec{b} + \vec{c})}{4} - \vec{b}}{q-1}$. (ii)



Comparing P.V. of F in (i) and (ii), we have

$$\frac{p\vec{c}}{p+1} = \frac{q\frac{(\vec{b} + \vec{c})}{4} - \vec{b}}{q-1}$$

Since vectors $\vec{b}$ and $\vec{c}$ are independent, we have

$$\frac{p}{p+1} = \frac{q}{4(q-1)} \text{ and } \frac{q-4}{4(q-1)} = 0$$

$\Rightarrow$ $p = 1/4$ and $q = 4$

$\Rightarrow$ $AF : FC = 1:2$

ILLUSTRATION 2.42

Prove that the necessary and sufficient condition for any four points in three-dimensional space to be coplanar is that there exists a linear relation connecting their position vectors such that the algebraic sum of the coefficients (not all zero) in it is zero.

Sol. Let us assume that the points A, B, C and D whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, are coplanar. In that case the lines AB and CD will intersect at some point P (it being assumed that AB and CD are not parallel, and if they are, then we will choose any other pair of non-parallel lines formed by the given points). If P divides AB in the ratio $q : p$ and CD in the ratio $n : m$, then the position vector of P written from AB and CD is

$$\frac{p\vec{a}+q\vec{b}}{p+q}=\frac{m\vec{c}+n\vec{d}}{m+n}$$

or $$\frac{p}{p+q}\vec{a}+\frac{q}{p+q}\vec{b}-\frac{m}{m+n}\vec{c}-\frac{n}{m+n}\vec{d}=\vec{0}$$

or $$L\vec{a}+M\vec{b}+N\vec{c}+P\vec{d}=\vec{0}$$

where $$L+M+N+P=\frac{p}{p+q}+\frac{q}{p+q}-\frac{m}{m+n}-\frac{n}{m+n}=1-1=0$$

Hence, the condition is necessary.

Converse: Let $l\vec{a}+m\vec{b}+n\vec{c}+p\vec{d}=\vec{0}$ (i)

where $l+m+n+p=0$

We will show that the points A, B, C and D are coplanar.

Now of the three scalars $l+m$, $l+n$ and $l+p$, one at least is not zero, because if all of them are zero, then

$$l+m=0, l+n=0, l+p=0$$

$\therefore$ $$m=n=p=-l$$

Hence, $l+m+n+p=0$ or $l-3l=0$ or $l=0$

Hence, $m=n=p=-l=0$

Thus, $l=0, m=0, n=0, p=0$, which is against the hypothesis.

Let us suppose that $l+m$ is not zero.

$$l+m=-(n+p)\neq 0,$$ [From (i)] (ii)

Also from the given relation, we have

$$l\vec{a}+m\vec{b}=-(n\vec{c}+p\vec{d})$$

or $$\frac{l\vec{a}+m\vec{b}}{l+m}=\frac{n\vec{c}+p\vec{d}}{n+p}$$ [From (ii)] (iii)

The L.H.S. represents a point which divides AB in the ratio $m : l$ and the R.H.S. represents a point which divides CD in the ratio $p : n$. These points being the same, it follows that a point on AB is the same as a point on CD, showing that the lines AB and CD intersect. Hence, the four points A, B, C and D are coplanar.

ILLUSTRATION 2.43

i. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, prove that vectors $3\vec{a}-7\vec{b}-4\vec{c}$, $3\vec{a}-2\vec{b}+\vec{c}$ and $\vec{a}+\vec{b}+2\vec{c}$ are coplanar.

ii. If the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar, then prove that $a=4$.

Sol.

i. If the given vectors are coplanar, then we should be able to express one of them as a linear combination of the other two.

Let us assume that $3\vec{a}-7\vec{b}-4\vec{c}=x(3\vec{a}-2\vec{b}+\vec{c})+y(\vec{a}+\vec{b}+2\vec{c})$,

where x and y are scalars. Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar, equating the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$$3x+y=3,\ -2x+y=-7,\ x+2y=-4$$

Solving the first two, we find that $x=2$ and $y=-3$. These values of x and y satisfy the third equation as well.

Hence, the given vectors are coplanar.

ii. Given vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar. Then

$$\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & -3\\ 3 & a & 5\end{vmatrix}=0$$

or $\quad 3-7a+25=0$

or $\quad a=4$

ILLUSTRATION 2.44

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors, prove that the four points $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar.

Sol. Let the given points be A, B, C and D. If they are coplanar, then the three coterminous vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ should be coplanar.

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-\vec{a}-5\vec{b}+4\vec{c}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\vec{a}+\vec{b}-\vec{c}$$

and $$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=-\vec{a}-9\vec{b}+7\vec{c}$$

Since the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar, we must have $\begin{vmatrix}-1 & -5 & 4\\ 1 & 1 & -1\\ -1 & -9 & 7\end{vmatrix}=0$, which is true.

Hence proved.

ILLUSTRATION 2.45

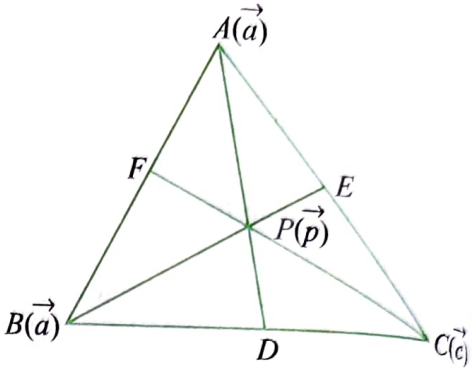
Let P be an interior point of a triangle ABC and AP, BP, CP meet the sides BC, CA, AB in D, E, F, respectively. Show that $\dfrac{AP}{PD}=\dfrac{AF}{FB}+\dfrac{AE}{EC}$.

Sol. Since A, B, C, P are co-planar, there exists four scalars x, y, z, w not all zero simultaneously such that

$$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{p} = 0$$

where $x + y + z + w = 0$

Also, $$\frac{x\vec{a} + w\vec{p}}{x + w} = \frac{y\vec{b} + z\vec{c}}{y + z}$$



Hence, $$\frac{AP}{PD} = -\frac{w}{x} - 1$$

Also $$\frac{x\vec{a} + y\vec{b}}{x + y} = \frac{z\vec{c} + w\vec{p}}{z + w}$$

$\Rightarrow$ $$\frac{AF}{FB} = \frac{y}{x}$$

Similarly, $$\frac{AE}{EC} = \frac{z}{x}$$

Thus, to show that $-\frac{w}{x} - 1 = \frac{y}{x} + \frac{z}{x}$

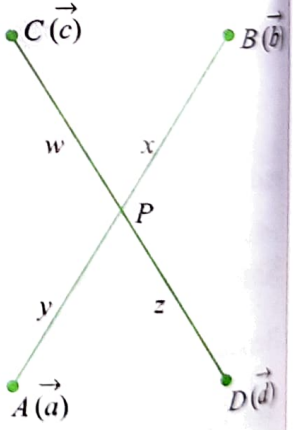
$\Rightarrow$ $x + y + z + w = 0$ which is true.

Hence proved.

ILLUSTRATION 2.46

Points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are related as $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = 0$ and $x + y + z + w = 0$, where x, y, z and w are scalars (sum of any two of x, y, z and w is not zero). Prove that if A, B, C and D are concyclic, then $|xy|\,|\vec{a} - \vec{b}|^2 = |wz|\,|\vec{c} - \vec{d}|^2$.

Sol. From the given conditions, it is clear that points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ and $D(\vec{d})$ are coplanar.



Now, A, B, C and D are concyclic. Therefore,

$$AP \times BP = CP \times DP$$

$$\left|\frac{y}{x+y}\right||\vec{a} - \vec{b}|\left|\frac{x}{x+y}\right||\vec{a} - \vec{b}| = \left|\frac{w}{w+z}\right||\vec{c} - \vec{d}|\left|\frac{z}{w+z}\right||\vec{c} - \vec{d}|$$

$$|xy|\,|\vec{a} - \vec{b}|^2 = |wz|\,|\vec{c} - \vec{d}|^2$$

CONCEPT APPLICATION EXERCISE 2.2

1. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vectors in three-dimensional space with the same initial point and such that $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$, show that terminals A, B, C and D of these vectors are coplanar. Find the point at which AC and BD meet. Find the ratio in which P divides AC and BD.

2. Show that the vectors $2\vec{a} - \vec{b} + 3\vec{c}, \vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$ are non-coplanar vectors (where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors).

3. Examine the following vectors for linear independence:

   i. $\vec{i} + \vec{j} + \vec{k}, 2\vec{i} + 3\vec{j} - \vec{k}, -\vec{i} - 2\vec{j} + 2\vec{k}$

   ii. $3\vec{i} + \vec{j} - \vec{k}, 2\vec{i} - \vec{j} + 7\vec{k}, 7\vec{i} - \vec{j} + 13\vec{k}$

4. If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A} = (p + 4q)\vec{a} + (2p + q + 1)\vec{b}$ and $\vec{B} = (-2p + q + 2)\vec{a} + (2p - 3q - 1)\vec{b}$, and if $3\vec{A} = 2\vec{B}$, then determine p and q.

5. If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three non-coplanar vectors, then prove that points $l_1\vec{a} + m_1\vec{b} + n_1\vec{c}$, $l_2\vec{a} + m_2\vec{b} + n_2\vec{c}, l_3\vec{a} + m_3\vec{b} + n_3\vec{c}, l_4\vec{a} + m_4\vec{b} + n_4\vec{c}$ are coplanar if $\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$.

6. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-zero, non-coplanar vectors, then find the linear relation between the following four vectors: $\vec{a} - 2\vec{b} + 3\vec{c}, 2\vec{a} - 3\vec{b} + 4\vec{c}, 3\vec{a} - 4\vec{b} + 5\vec{c}, 7\vec{a} - 11\vec{b} + 15\vec{c}$.

7. Let a, b, c be distinct non-negative numbers and the vectors $a\hat{i} + a\hat{j} + c\hat{k}, \hat{i} + \hat{k}, c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane, then prove that the quadratic equation $ax^2 + 2cx + b = 0$ has equal roots.

ANSWERS

1. AC and BD intersect at P whose P.V. is $\frac{3\vec{a} + \vec{c}}{4}$; P divides AC in the ratio 3 : 1 and BD in the ratio 1 : 1.
3. (i) Vectors are non-coplanar and linearly independent.
   (ii) Vectors are coplanar and linearly dependent
4. $p = 2$ and $q = -1$
6. $7\vec{a} - 11\vec{b} + 15\vec{c} = \vec{a} - 2\vec{b} + 3\vec{c} + 3(2\vec{a} - 3\vec{b} + 4\vec{c}) + 0(3\vec{a} - 4\vec{b} + 5\vec{c})$

Solved Examples

EXAMPLE 2.1

Find all values of λ such that $x, y, z \neq (0, 0, 0)$ and $(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z = \lambda(x\hat{i}+y\hat{j}+z\hat{k})$, where, $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors along the coordinate axes.

Sol. $(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z = \lambda(x\hat{i}+y\hat{j}+z\hat{k})$

Comparing coefficient of $\hat{i}$, $x+3y-4z=\lambda x$ (i)

$\Rightarrow$ $(1-\lambda)x+3y-4z=0$

Comparing coefficient of $\hat{j}$, $x-3y+5z=\lambda y$ (ii)

$\Rightarrow$ $x-(3+\lambda)y+5z=0$

Comparing coefficient of $\hat{k}$, $3x+y+0z=\lambda z$ (iii)

$3x+y-\lambda z=0$

All the above three equations are satisfied for x, y and z not all zero if

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

or $(1-\lambda)[3\lambda+\lambda^2-5]-3[-\lambda-15]-4[1+9+3\lambda]=0$

or $\lambda^3+2\lambda^2+\lambda=0$

or $\lambda(\lambda+1)^2=0$

or $\lambda=0,-1$

EXAMPLE 2.2

A vector has components A_1, A_2 and A_3 in a right-handed rectangular Cartesian coordinate system $OXYZ$. The coordinate system is rotated about the x-axis through an angle $\pi/2$. Find the components of A in the new coordinate system in terms of A_1, A_2 and A_3.

Sol. Since vector $\vec{A}$ has components A_1, A_2 and A_3, in the coordinate system $OXYZ$,

$$\vec{A} = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$$

When given system is rotated through $\pi/2$, the new x-axis is along the old y-axis and the new y-axis is along the old negative x-axis; z remains same as before.

Hence, the components of A in the new system are A_2, $-A_1$ and A_3.

Therefore, $\vec{A}$ becomes $A_2\hat{i}-A_1\hat{j}+A_3\hat{k}$.

EXAMPLE 2.3

The position vectors of the vertices A, B and C of a triangle are $\hat{i}+\hat{j}$, $\hat{j}+\hat{k}$ and $\hat{i}+\hat{k}$, respectively. Find a unit vector $\hat{r}$ lying in the plane of ABC and perpendicular to IA, where I is the incentre of the triangle.

Sol. Since $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$, the incentre is same as the circumcentre, and hence IA is perpendicular to BC. Therefore, $\vec{r}$ is parallel to BC.

$$\vec{r} = \lambda(\hat{i} - \hat{j})$$

Hence, unit vector $\vec{r} = \pm\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$

EXAMPLE 2.4

A ship is sailing towards the north at a speed of 1.25 m/s. The current is taking it towards the east at the rate of 1 m/s and a sailor is climbing a vertical pole on the ship at the rate of 0.5 m/s. Find the velocity of the sailor in space.

Sol. We take the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ parallel to the east, north and vertically upwards in the direction of pole, respectively. Then the velocity vectors of the current, ship and the sailor are, respectively, $\hat{i}$, $1.25\,\hat{j}$ and $0.5\,\hat{k}$. Velocity $\vec{v}$ of the sailor in space is the resultant of these vectors. Hence,

$$\vec{v} = \hat{i} + 1.25\,\hat{j} + 0.5\,\hat{k}$$

Then
$$|\vec{v}| = \sqrt{1 + (1.25)^2 + (0\cdot 5)^2}$$
$$= \sqrt{1 + 1.5625 + .25}$$
$$= \sqrt{2.8125} = 1.677 \text{ m/s}$$

EXAMPLE 2.5

Given four points P_1, P_2, P_3 and P_4 on the coordinate plane with origin O which satisfy the condition $\overrightarrow{OP}_{n-1} + \overrightarrow{OP}_{n+1} = \frac{3}{2}\overrightarrow{OP}_n$.

i. If P_1 and P_2 lie on the curve $xy = 1$, then prove that P_3 does not lie on the curve.

ii. If P_1, P_2 and P_3 lie on the circle $x^2 + y^2 = 1$, then prove that P_4 also lies on this circle.

Sol. **(i)** Put $n = 2$ in $\overrightarrow{OP}_{n-1} + \overrightarrow{OP}_{n+1} = \frac{3}{2}\overrightarrow{OP}_n$

$$\overrightarrow{OP}_3 = \frac{3}{2}\overrightarrow{OP}_2 - \overrightarrow{OP}_1 \qquad \text{(i)}$$

$$\left.\begin{aligned}\overrightarrow{OP_1} &= a\,\hat{i} + \frac{1}{a}\hat{j}\\ \overrightarrow{OP_2} &= b\,\hat{i} + \frac{1}{b}\hat{j}\end{aligned}\right\} ab \neq 0$$

$\therefore$
$$\overrightarrow{OP_3} = \frac{3}{2}\left(b\,\hat{i} + \frac{1}{b}\hat{j}\right) - \left(a\,\hat{i} + \frac{1}{a}\hat{j}\right)$$
$$= \left(\frac{3b}{2} - a\right)\hat{i} + \left(\frac{3}{2b} - \frac{1}{a}\right)\hat{j}$$

If P_3 lies on $xy = 1$, we have

$$\left(\frac{3b}{2} - a\right)\left(\frac{3}{2b} - \frac{1}{a}\right) = 1$$

or $(3b - 2a)(3a - 2b) = 4ab$

or $9ab - 6b^2 - 6a^2 + 4ab = 4ab$

or $2a^2 - 3ab + 2b^2 = 0$

which is not possible as Discriminant < 0 ($a = 0$ and $b = 0$ not possible)

(ii) $\overrightarrow{OP} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$

$\overrightarrow{OP_2} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$

$\therefore$ $\overrightarrow{OP_3} = \frac{3}{2}(\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) - (\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$

$$= \left(\frac{3}{2}\cos\beta - \cos\alpha\right)\hat{i} + \left(\frac{3}{2}\sin\beta - \sin\alpha\right)\hat{j}$$

Since P_3 lies on $x^2 + y^2 = 1$, we have

$$\left(\frac{3}{2}\cos\beta - \cos\alpha\right)^2 + \left(\frac{3}{2}\sin\beta - \sin\alpha\right)^2 = 1$$

or $\frac{9}{4} + 1 - 3(\cos\beta\cos\alpha + \sin\beta\sin\alpha) = 1$

or $\frac{9}{4} - 3\cos(\beta - \alpha) = 0 \Rightarrow \cos(\beta - \alpha) = \frac{3}{4}$ (ii

Put $n = 3$ in the given relation.

$$\overrightarrow{OP_2} + \overrightarrow{OP_4} = \frac{3}{2}\overrightarrow{OP_3}$$

or $\overrightarrow{OP_4} = \frac{3}{2}\overrightarrow{OP_3} - \overrightarrow{OP_2}$

$$= \frac{3}{2}\left(\frac{3}{2}\overrightarrow{OP_2} - \overrightarrow{OP_1}\right) - \overrightarrow{OP_2}$$

$$= \frac{5}{4}\overrightarrow{OP_2} - \frac{3}{2}\overrightarrow{OP_1}$$

$$= \frac{5}{4}(\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) - \frac{3}{2}(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}),$$

which lies on $x^2 + y^2 = 1$

EXAMPLE 2.6

ABCD is a tetrahedron and *O* is any point. If the lines joining *O* to the vertices meet the opposite faces at *P*, *Q*, *R* and *S*, prove that $\frac{OP}{AP} + \frac{OQ}{BQ} + \frac{OR}{CR} + \frac{OS}{DS} = 1$.

Sol. Here $ABCD$ is a tetrahedron. Let O be the origin and the P.V. of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively.

We know that four linearly dependent vectors can be expressed as

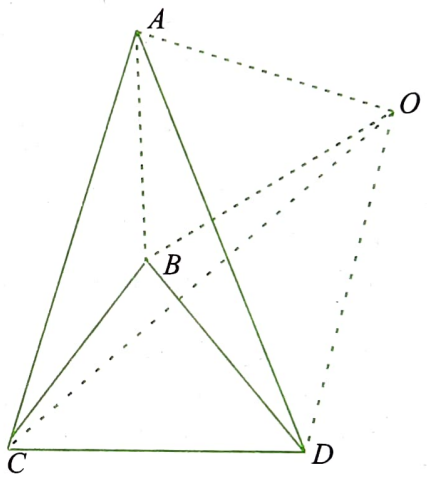
$$x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0 \qquad \text{(where } x, y, z \text{ and } t \text{ are scalars)}$$

or $$y\vec{b} + z\vec{c} + t\vec{d} = -x\vec{a}$$

or $$\frac{y\vec{b} + z\vec{c} + t\vec{d}}{y+z+t} = -\frac{x\vec{a}}{y+z+t}$$

where L.H.S. is P.V. of a point in the plane BCD and R.H.S. is a point on $\overrightarrow{AO}$.

Therefore, there must be a point common to both the plane and the straight line. That is



$$\overrightarrow{OP} = \frac{-x\vec{a}}{y+z+t}$$

But, $$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = -\frac{x\vec{a}}{y+z+t} - \vec{a}$$

$$= -\left(\frac{x+y+z+t}{y+z+t}\right)\vec{a}$$

$$\overrightarrow{OP} = \frac{x}{y+z+t}\left(\frac{y+z+t}{x+y+z+t}\right)\overrightarrow{AP}$$

$$= \left(\frac{x}{x+y+z+t}\right)\overrightarrow{AP}$$

$$\Rightarrow \quad \frac{OP}{AP} = \frac{x}{x+y+z+t}$$

Similarly, $\dfrac{OQ}{BQ} = \dfrac{y}{x+y+z+t}$

$$\frac{OR}{CR} = \frac{z}{x+y+z+t} \text{ and } \frac{OS}{DS} = \frac{t}{x+y+z+t}$$

$$\Rightarrow \quad \frac{OP}{AP} + \frac{OQ}{BQ} + \frac{OR}{CR} + \frac{OS}{DS} = 1$$

EXAMPLE 2.7

Prove, by vector method or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the midpoint of the parallel sides (you may assume that the trapezium is not a parallelogram.)

Sol. Let the P.V.s of the points A, B, C and D be $\vec{O}$, $B(\vec{b})$, $D(\vec{d})$ and $C(\vec{d} + t\vec{b})$

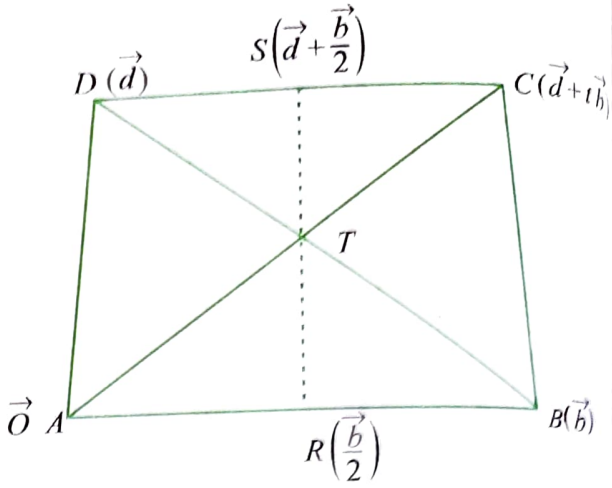
For any point $\vec{r}$ on $\overrightarrow{AC}$ and $\overrightarrow{BD}$, $\vec{r} = \lambda(\vec{d} + t\vec{b})$ and $\vec{r} = (1-\mu)\vec{b} + \mu\vec{d}$, respectively.

For the point of intersection, say T, compare the coefficients.

$$\lambda = \mu,\ t\lambda = 1 - \mu = 1 - \lambda \text{ or } (t+1)\lambda = 1$$

$$\therefore \quad \lambda = \frac{1}{t+1} = \mu$$

Therefore, $\vec{r}$ (position vector of T) $= \dfrac{\vec{d} + t\vec{b}}{t+1}$ (i)



Let R and S be the midpoints of the parallel sides AB and DC;

then R is $\dfrac{b}{2}$ and S is $d + t\dfrac{b}{2}$.

Let T divide SR in the ratio $m:1$.

Position vector of T is $\dfrac{m\dfrac{\vec{b}}{2} + \vec{d} + t\dfrac{\vec{b}}{2}}{m+1}$, which is equivalent to $\dfrac{\vec{d} + t\vec{b}}{t+1}$.

Comparing coefficients of $\vec{b}$ and $\vec{d}$,

$$\frac{1}{m+1} = \frac{1}{t+1} \text{ and } \frac{m+t}{2(m+1)} = \frac{t}{t+1}.$$

From the first relation, $m = t$, which satisfies the second relation. Hence proved.

EXAMPLE 2.8

Show, by vector method, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices.

Sol. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of A, B and C, respectively.

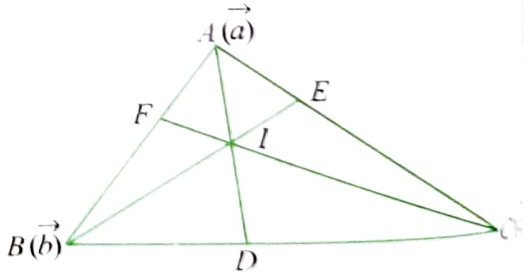
Let AD, BE and CF be the bisectors of $\angle A$, $\angle B$ and $\angle C$, respectively.

a, b and c are the lengths of sides BC, CA and AB, respectively.

Now AD divides BC in the ratio

$$BD : DC = AB : AC = c : b.$$

Hence, the position vector of D is $\vec{d} = \dfrac{b\vec{b} + c\vec{c}}{b+c}$.



Let I be the point of intersection of BE and AD.

Then in ΔABC, BI is bisector of $\angle B$. Therefore,

$$DI : IA = BD : BA$$

But $$\frac{BD}{DC} = \frac{c}{b} \text{ or } \frac{BD}{BD + DC} = \frac{c}{c+b}$$

or $$\frac{BD}{BC} = \frac{c}{c+b}$$

or $$BD = \frac{ac}{b+c}$$

$$\therefore \quad DI : IA = \frac{ac}{b+c} : c = a : (b+c)$$

$$\therefore \quad \text{P.V. of } I = \frac{\vec{a}\,a + \vec{d}(b+c)}{a+b+c}$$

$$= \frac{\vec{a}\,a + \left(\frac{\vec{b}\,b + \vec{c}\,c}{b+c}\right)(b+c)}{a+b+c}$$

$$= \frac{\vec{a}\,a + \vec{b}\,b + \vec{c}\,c}{a+b+c}$$

As P.V. of I is symmetrical in $\vec{a}, \vec{b}, \vec{c}$ and a, b, c, it must lie on CF as well.

EXAMPLE 2.9

A pyramid with vertex at point P has a regular hexagonal base $ABCDEF$. Position vectors of points A and B are $\hat{i}$ and $\hat{i}+2\hat{j}$, respectively. The centre of the base has the position vector $\hat{i}+\hat{j}+\sqrt{3}\hat{k}$. Altitude drawn from P on the base meets the diagonal AD at point G. Find all possible position vectors of G. It is given that the volume of the pyramid is $6\sqrt{3}$ cubic units and AP is 5 units.

Sol. Let the centre of the base be (0). Therefore,

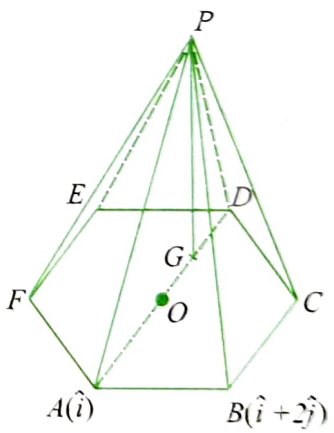
$$|\overrightarrow{AB}| = 2$$

$$\Delta OAB = \frac{1}{4} \times 4 \times \sqrt{3} = \sqrt{3}$$

Base area $= 6\sqrt{3}$ sq. units

Let height of the pyramid be h. Therefore,

$$\frac{1}{3} \times 6\sqrt{3}h = 6\sqrt{3} \text{ or } h = 3 \text{ units}$$



It is given that $|\overrightarrow{AP}| = 5$. Therefore,

$$AG = \sqrt{25-9} = 4 \text{ units}$$

$$\Rightarrow \quad |\overrightarrow{AG}| = 4 \text{ units}$$

Now $|\overrightarrow{AG}|$ and $|\overrightarrow{AO}|$ are collinear. Therefore,

$$\overrightarrow{AG} = \lambda\,\overrightarrow{AO} \Rightarrow |\overrightarrow{AG}| = |\lambda||\overrightarrow{AO}|$$

or $\quad 2|\lambda| = 4 \text{ or } |\lambda| = 2$

$$\Rightarrow \quad \overrightarrow{AG} = \pm 2(\hat{i}+\hat{j}+\sqrt{3}\hat{k})$$

$$\Rightarrow \quad \vec{OG} = \pm 2(\hat{i}+\hat{j}+\sqrt{3}\hat{k}) + \hat{i}$$

$$= -(\hat{i}+2\hat{j}+2\sqrt{3}\hat{k}),\ 3\hat{i}+2\hat{j}+2\sqrt{3}\hat{k}$$

EXAMPLE 2.10

A straight line L cuts the lines AB, AC and AD of a parallelogram $ABCD$ at points B_1, C_1 and D_1, respectively. If $\overrightarrow{AB_1} = \lambda_1 \overrightarrow{AB}$, $\overrightarrow{AD_1} = \lambda_2 \overrightarrow{AD}$ and $\overrightarrow{AC_1} = \lambda_3 \overrightarrow{AC}$, then prove that $\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$.

Sol. Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$; then $\overrightarrow{AC} = \vec{a} + \vec{b}$.

Given $\overrightarrow{AB_1} = \lambda_1 \vec{a}$, $\overrightarrow{AD_1} = \lambda_2 \vec{b}$, $\overrightarrow{AC_1} = \lambda_3(\vec{a} + \vec{b})$

$$\overrightarrow{B_1D_1} = \overrightarrow{AD_1} - \overrightarrow{AB_1} = \lambda_2 \vec{b} - \lambda_1 \vec{a}$$

Since vectors $\overrightarrow{D_1C_1}$ and $\overrightarrow{B_1D_1}$ are collinear, we have

$$\overrightarrow{D_1C_1} = k\overrightarrow{B_1D_1} \text{ for some } k \in R$$

$$\Rightarrow \quad \overrightarrow{AC_1} - \overrightarrow{AD_1} = k\overrightarrow{B_1D_1}$$

$$\Rightarrow \quad \lambda_3(\vec{a} + \vec{b}) - \lambda_2 \vec{b} = k(\lambda_2 \vec{b} - \lambda_1 \vec{a})$$

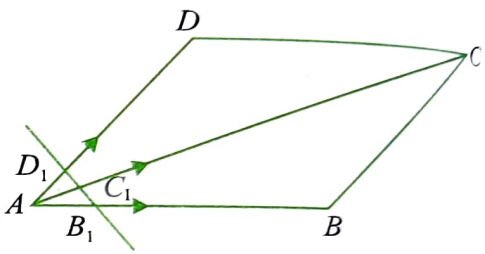
or $$\lambda_3 \vec{a} + (\lambda_3 - \lambda_2)\vec{b} = k\lambda_2 \vec{b} - k\lambda_1 \vec{a}$$

Hence, $\lambda_3 = -k\lambda_1$ and $\lambda_3 - \lambda_2 = k\lambda_2$

$$\Rightarrow \quad k = -\frac{\lambda_3}{\lambda_1} = \frac{\lambda_3 - \lambda_2}{\lambda_2}$$

or $$\lambda_1\lambda_2 = \lambda_1\lambda_3 + \lambda_2\lambda_3$$

or $$\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$



EXAMPLE 2.11

The position vectors of the points P and Q are $5\hat{i} + 7\hat{j} - 2\hat{k}$ and $-3\hat{i} + 3\hat{j} + 6\hat{k}$, respectively. Vector $\vec{A} = 3\hat{i} - \hat{j} + \hat{k}$ passes through point P and vector $\vec{B} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ passes through point Q. A third vector $2\hat{i} + 7\hat{j} - 5\hat{k}$ intersects vectors A and B. Find the position vectors of points of intersection.

Sol. Let vector $2\hat{i} + 7\hat{j} - 5\hat{k}$ intersect vectors $\vec{A}$ and $\vec{B}$ at points L and M, respectively, which have to be determined. Take them to be (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively.

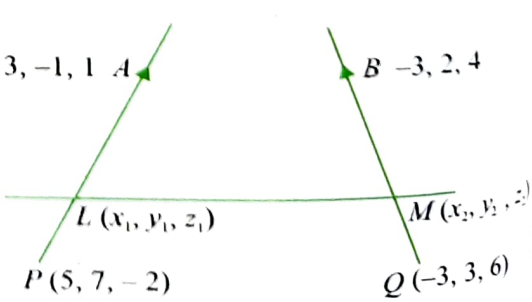
PL is collinear with vector $\vec{A}$. Therefore,

$$\overrightarrow{PL} = \lambda \vec{A}$$

Comparing the coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$\frac{x_1 - 5}{3} = \frac{y_1 - 7}{-1} = \frac{z_1 + 2}{1} = \lambda \text{ (say)}$$

L is $3\lambda + 5, -\lambda + 7, \lambda - 2$



Similarly, $\overrightarrow{QM} = \mu \vec{B}$. Therefore,

$$\frac{x_3 + 3}{-3} = \frac{y_2 - 3}{2} = \frac{z_2 - 6}{4} = \mu \text{ (say)}$$

Therefore, M is $-3\mu - 3, 2\mu + 3, 4\mu + 6$

Again LM is collinear with vector $2\hat{i} + 7\hat{j} - 5\hat{k}$. Therefore,

$$\frac{x_2 - x_1}{2} = \frac{y_2 - y_1}{7} = \frac{z_2 - z_1}{-5} = \nu \text{ (say)}$$

$$\frac{-3\mu - 3\lambda - 8}{2} = \frac{2\mu + \lambda - 4}{7} = \frac{4\mu - \lambda + 8}{-5} = \nu$$

$$3\mu + 3\lambda + 2\nu = -8$$

$$2\mu + \lambda - 7\nu = 4$$

$$4\mu - \lambda + 5\nu = -8$$

Solving, we get

$$\lambda = \mu = \nu = -1$$

Therefore, point L is $(2, 8, -3)$ or $2\hat{i} + 8\hat{j} - 3\hat{k}$

and M is $(0, 1, 2)$ or $\hat{j} + 2\hat{k}$

EXAMPLE 2.12

Show that $x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$ are non-coplanar if $|x_1| > |y_1| + |z_1|$, $|y_2| > |x_2| + |z_2|$ and $|z_3| > |x_3| + |y_3|$.

Sol. If the given vectors are coplanar, then

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

or the set of equations

$$x_1x + y_1y + z_1z = 0,$$

and

$$x_2x + y_2y + z_2z = 0$$

$$x_3x + y_3y + z_3z = 0 \text{ has a non-trivial solution.}$$

Let the given set has a non-trivial solution x, y, z without the loss of generality; we can assume that $x \geq y \geq z$.

For the given equation $x_1x + y_1y + z_1z = 0$, we have $x_1x = -y_1y - z_1z$. Therefore,

$$|x_1x| = |y_1y + z_1z| \leq |y_1y| + |z_1z|$$

$$\Rightarrow \quad |x_1x| \leq |y_1x| + |z_1x|$$

$$\Rightarrow \quad |x_1| \leq |y_1| + |z_1|,$$

which is a contradiction to the given inequality, i.e.,

$$|x_1| > |y_1| + |z_1|.$$

Similarly, the other inequalities rule out the possibility of a non-trivial solution.

Hence, the given equation has only a trivial solution. Hence, the given vectors are non-coplanar.

EXAMPLE 2.13

If $\vec{A}$ and $\vec{B}$ are two vectors and k any scalar quantity greater than zero, then prove that $|\vec{A}+\vec{B}|^2 \le (1+k)|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2$.

Sol. We know:

$$(1+k)|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2$$

$$= |\vec{A}|^2 + k|\vec{A}|^2 + |\vec{B}|^2 + \frac{1}{k}|\vec{B}|^2 \qquad \text{(i)}$$

Also, $$k|\vec{A}|^2 + \frac{1}{k}|\vec{B}|^2 \ge 2\left(k + |\vec{A}|^2 \cdot \frac{1}{k}|\vec{B}|^2\right)^{1/2}$$

$$= 2|\vec{A}|\cdot|\vec{B}| \qquad \text{(ii)}$$

(Since arithmetic mean ≥ geometric mean)

$\therefore$ $$(1+k)|\vec{A}|^2 + \left(1+\frac{1}{k}\right)|\vec{B}|^2$$

$$\ge |\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}|\cdot|\vec{B}| = (|\vec{A}|+|\vec{B}|)^2 \qquad \text{[Using (i) and (ii)]}$$

And also $|\vec{A}|+|\vec{B}| \ge |\vec{A}+\vec{B}|$

Hence, $(1+k)|\vec{A}|^2 + \left(1+\frac{1}{k}\right)^2 |\vec{B}|^2 \ge |\vec{A}+\vec{B}|^2$

EXAMPLE 2.14

Consider the vectors $\hat{i} + \cos(\beta-\alpha)\,\hat{j} + \cos(\gamma-\alpha)\,\hat{k}$, $\cos(\alpha-\beta)\,\hat{i} + \hat{j} + \cos(\gamma-\beta)\,\hat{k}$ and $\cos(\alpha-\gamma)\,\hat{i} + \cos(\beta-\gamma)\,\hat{j} + a\hat{k}$, where α, β and γ are different angles. If these vectors are coplanar, show that a is independent of α, β and γ.

Sol. Since the vectors are coplanar, we have

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & a \end{vmatrix} = 0$$

$$\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & a-1 \end{vmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $a = 1$

EXAMPLE 2.15

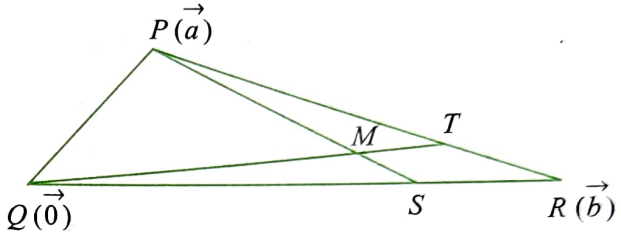
In a triangle PQR, S and T are points on QR and PR, respectively, such that $QS = 3SR$ and $PT = 4TR$. Let M be the point of intersection of PS and QT. Determine the ratio $QM : MT$ using the vector method.

Sol. Let $QM : MT = \lambda : 1$ and $PM : MS = \mu : 1$

and $\overrightarrow{QP} = \vec{a}, \overrightarrow{QR} = \vec{b}$

$\Rightarrow$ $\overrightarrow{QT} = \dfrac{4\vec{b} + \vec{a}}{5}$

and $\overrightarrow{QM} = \dfrac{\lambda}{\lambda + 1}\left(\dfrac{4\vec{b} + \vec{a}}{5}\right)$ (i)



$\overrightarrow{QS} = \dfrac{3}{4}\vec{b}, \overrightarrow{QM} = \dfrac{\mu\left(\frac{3}{4}\vec{b}\right) + \vec{a}}{\mu + 1}$ (ii)

From (i) and (ii), we have

$$\frac{1}{\mu + 1} = \frac{\lambda}{5(\lambda + 1)} \text{ and } \frac{4\lambda}{5(\lambda + 1)} = \frac{3\mu}{4(\mu + 1)}$$

$\Rightarrow$ $\lambda = 15/4$ and $\mu = 16/3$

$\therefore$ $QM : MT = 15 : 4$

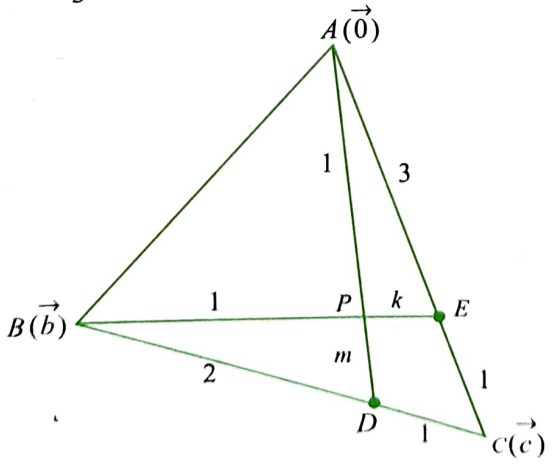
EXAMPLE 2.16

In a triangle ABC, D and E are points on BC and AC, respectively, such that $BD = 2DC$ and $AE = 3EC$. Let P be the point of intersection of AD and BE. Find BP/PE using the vector method.

Sol. Let the vertices of the triangle be $A(\vec{0})$, $B(\vec{b})$ and $C(\vec{c})$.

Given that D divides BC in the ratio $2 : 1$.

Therefore, position vector of D is $\dfrac{\vec{b} + 2\vec{c}}{3}$.



E divides AC in the ratio 3 : 1.

Therefore, position vector of E is $\dfrac{\vec{0}+3\vec{c}}{4}=\dfrac{3\vec{c}}{4}$.

Let point of intersection P of AD and BE divide BE in the ratio $1 : k$ and AD in the ratio $1 : m$. Then position vectors of P in these two cases are $\dfrac{k\vec{b}+1(3\vec{c}/4)}{k+1}$ and $\dfrac{m\vec{0}+m((\vec{b}+2\vec{c})/3)}{m+1}$, respectively.

Equating the position vectors of P in these two cases, we get

$$\frac{k\vec{b}}{k+1}+\frac{3\vec{c}}{4(k+1)}=\frac{m\vec{b}}{3(m+1)}+\frac{2m\vec{c}}{3(m+1)}$$

$$\Rightarrow \quad \frac{k}{k+1}=\frac{m}{3(m+1)} \text{ and } \frac{3}{4(k+1)}=\frac{2m}{3(m+1)}$$

Dividing, we have $\dfrac{4k}{3}=\dfrac{1}{2}$ or $k=\dfrac{3}{8}$

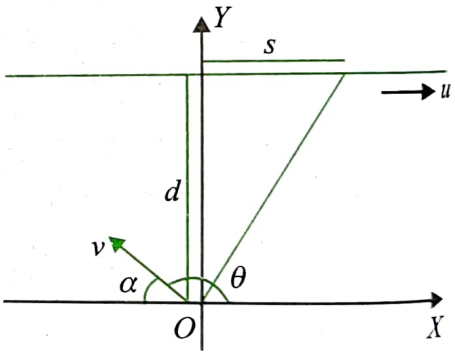
Required ratio is 8 : 3.

EXAMPLE 2.17

A boat moves in still water with a velocity which is k times less than the river flow velocity. Find the angle to the stream direction at which the boat should be rowed to minimize drifting.

Sol. Let the flow velocity of river be u and the velocity of boat in still water be v. Thus,

$$v=\frac{u}{K}$$



Also, let the boat move at an angle θ with the direction of stream.

Now the velocity of boat in the river is the vector resultant of the velocity of boat and flow velocity of river, which can be written as

$$\vec{v}_B=(u-v\cos\alpha)\hat{i}+(u\sin\alpha)\hat{j}$$

$$=(u+v\cos\theta)\hat{i}+(u\sin\theta)\hat{j}$$

Hence, the time taken to cross the river $=\dfrac{d}{u\sin\theta}$ (d = width of the river)

Thus, the drift $s=(u+v\cos\theta)\cdot d$

or $$s=d\left(\operatorname{cosec}\theta+\frac{v}{u}\cot\theta\right)$$

or $$\frac{ds}{d\theta}=d\left(\operatorname{cosec}\theta\cot\theta-\frac{v}{\mu}\operatorname{cosec}^2\theta\right)=0$$

or $$\frac{v}{u}\operatorname{cosec}^2\theta=\operatorname{cosec}\theta\cot\theta$$

or $$\cos\theta=\frac{1}{k} \text{ or } \theta=\cos^{-1}\left(\frac{1}{k}\right)$$

EXAMPLE 2.18

If D, E and F are three points on the sides BC, CA and AB, respectively, of a triangle ABC such that the lines AD, BE and CF are concurrent, then show that

$$\frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF}=-1.$$

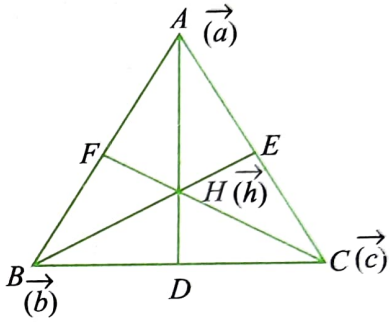
Sol. $x\vec{a}+y\vec{b}+z\vec{c}+t\vec{h}=0$ such that

$$x+y+z+t=0 \qquad \text{(i)}$$

$$x\vec{a}+y\vec{b}=-(z\vec{c}+t\vec{h})$$

and $$x+y=-(z+t)$$

$$\therefore \quad \frac{x\vec{a}+y\vec{b}}{x+y}=\frac{z\vec{c}+t\vec{h}}{z+t} \qquad \text{(ii)}$$



Position vector of $F=\dfrac{x\vec{a}+y\vec{b}}{x+y}$

Hence, F divides AB in the ratio y/x.

$$\frac{AF}{FB}=\frac{y}{x}$$

Similarly, $\dfrac{BD}{CD}=\dfrac{z}{y}$ and $\dfrac{CE}{AE}=\dfrac{x}{z}$

$$\Rightarrow \quad \frac{AF}{FB}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=-1$$

EXAMPLE 2.19

In a quadrilateral $PQRS$, $\overrightarrow{PQ}=\vec{a}$, $\overrightarrow{QR}=\vec{b}$, $\overrightarrow{SP}=\vec{a}-\vec{b}$, M is the midpoint of $\overrightarrow{QR}$ and X is a point on SM such that $SX=\dfrac{4}{5}SM$. Prove that P, X and R are collinear.

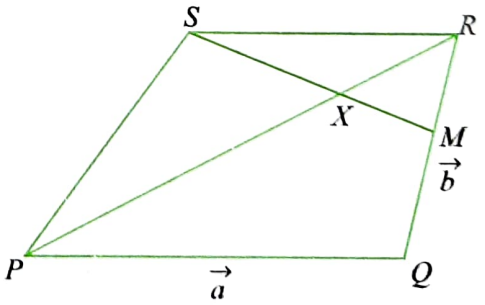
Sol. $\overrightarrow{OM}=\dfrac{\vec{b}}{2} \Rightarrow \overrightarrow{PM}=\vec{a}+\dfrac{\vec{b}}{2}$

$$\overrightarrow{SM}=\overrightarrow{PM}-\overrightarrow{PS}=2\vec{a}-\frac{1}{2}\vec{b}$$

$$\overrightarrow{SX}=\frac{4}{5}\overrightarrow{SM}=\frac{8}{5}\vec{a}-\frac{2}{5}\vec{b}$$

$$\overrightarrow{PX}=\overrightarrow{PS}+\overrightarrow{SX}$$

$$=-\vec{a}+\vec{b}+\frac{8}{5}\vec{a}-\frac{2}{5}\vec{b}=\frac{3}{5}(\vec{a}+\vec{b})$$



Also $\overrightarrow{PR}=\overrightarrow{PQ}+\overrightarrow{QR}=\vec{a}+\vec{b}=\dfrac{5}{3}\overrightarrow{PX}$

Hence P, X and R are collinear.

EXAMPLE 2.20

Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j}, t \in [0, 1]$, where f_1, f_2, g_1, g_2 are continuous functions. If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors for all and $\vec{A}(0) = 2\hat{i} + 3\hat{j}$, $\vec{A}(1) = 6\hat{i} + 2\hat{j}$, $\vec{B}(0) = 3\hat{i} + 2\hat{j}$ and $\vec{B}(1) = 2\hat{i} + 6\hat{j}$, then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t.

Sol. $\vec{A}(t)$ is parallel to $\vec{B}(t)$ for some $t \in [0, 1]$ if and only if $\frac{f_1(t)}{g_1(t)} = \frac{f_2(t)}{g_2(t)}$ for some $t \in [0, 1]$

or $f_1(t) \cdot g_2(t) = f_2(t) g_1(t)$ for some $t \in [0, 1]$

Let $h(t) = f_1(t) \cdot g_2(t) - f_2(t) \cdot g_1(t)$

$h(0) = f_1(0) \cdot g_2(0) - f_2(0) \cdot g_1(0)$

$= 2 \times 2 - 3 \times 3 = -5 < 0$

$h(1) = f_1(1) \cdot g_2(1) - f_2(1) \cdot g_1(1)$

$= 6 \times 6 - 2 \times 2 = 32 > 0$

Since h is a continuous function, and $h(0) \cdot h(1) < 0$, there are some $t \in [0, 1]$ for which $h(t) = 0$, i.e., $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors for this t.

Exercises

Single Correct Answer Type

1. Four non-zero vectors will always be
   (1) linearly dependent
   (2) linearly independent
   (3) either (1) or (2)
   (4) none of these

2. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three units vectors such that $3\vec{a}+4\vec{b}+5\vec{c}=0$. Then which of the following statements is true?
   (1) $\vec{a}$ is parallel to $\vec{b}$
   (2) $\vec{a}$ is perpendicular to $\vec{b}$
   (3) $\vec{a}$ is neither parallel nor perpendicular to $\vec{b}$
   (4) none of these

3. Let ABC be a triangle, the position vectors of whose vertices are respectively $\hat{i}+2\hat{j}+4\hat{k}$, $-2\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}+4\hat{j}-3\hat{k}$. Then ΔABC is
   (1) isosceles (2) equilateral (3) right angled (4) none of these

4. If $|\vec{a}+\vec{b}|<|\vec{a}-\vec{b}|$, then the angle between $\vec{a}$ and $\vec{b}$ can lie in the interval
   (1) $(-\pi/2, \pi/2)$ (2) $(0, \pi)$ (3) $(\pi/2, 3\pi/2)$ (4) $(0, 2\pi)$

5. A point O is the centre of a circle circumscribed about a triangle ABC. Then $\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C$ is equal to
   (1) $(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})\sin 2A$
   (2) $3\overrightarrow{OG}$, where G is the centroid of triangle ABC
   (3) $\vec{0}$
   (4) none of these

6. If G is the centroid of a triangle ABC, then $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}$ is equal to
   (1) $\vec{0}$ (2) $3\overrightarrow{GA}$ (3) $3\overrightarrow{GB}$ (4) $3\overrightarrow{GC}$

7. If $\vec{a}$ is a non-zero vector of modulus a and m is a non-zero scalar, then $m\vec{a}$ is a unit vector if
   (1) $m=\pm 1$ (2) $a=|m|$ (3) $a=1/|m|$ (4) $a=1/m$

8. $ABCD$ a parallelogram, and A_1 and B_1 are the midpoints of sides BC and CD, respectively. If $\overrightarrow{AA_1}+\overrightarrow{AB_1}=\lambda\overrightarrow{AC}$, then λ is equal to
   (1) $\frac{1}{2}$ (2) 1 (3) $\frac{3}{2}$ (4) 2

9. The position vectors of the points P and Q with respect to the origin O are $\vec{a}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}-2\hat{k}$, respectively. If M is a point on PQ, such that OM is the bisector of POQ, then $\overrightarrow{OM}$ is
   (1) $2(\hat{i}-\hat{j}+\hat{k})$ (2) $2\hat{i}+\hat{j}-2\hat{k}$ (3) $2(-\hat{i}+\hat{j}-\hat{k})$ (4) $2(\hat{i}+\hat{j}+\hat{k})$

10. $ABCD$ is a quadrilateral. E is the point of intersection of the line joining the midpoints of the opposite sides. If O is any point and $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = x\overrightarrow{OE}$, then x is equal to

(1) 3 (2) 9 (3) 7 (4) 4

11. If vectors $\overrightarrow{AB} = -3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are the sides of a ΔABC, then the length of the median through A is

(1) $\sqrt{14}$ (2) $\sqrt{18}$ (3) $\sqrt{29}$ (4) 5

12. A, B, C and D have position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, such that $\vec{a} - \vec{b} = 2(\vec{d} - \vec{c})$. Then

(1) AB and CD bisect each other
(2) BD and AC bisect each other
(3) AB and CD trisect each other
(4) BD and AC trisect each other

13. If $\vec{a}$ and $\vec{b}$ are two unit vectors and θ is the angle between them, then the unit vector along the angular bisector of $\vec{a}$ and $\vec{b}$ will be given by

(1) $\dfrac{\vec{a} - \vec{b}}{2\cos(\theta/2)}$ (2) $\dfrac{\vec{a} + \vec{b}}{2\cos(\theta/2)}$ (3) $\dfrac{\vec{a} - \vec{b}}{\cos(\theta/2)}$ (4) none of these

14. Let us define the length of a vector $a\hat{i} + b\hat{j} + c\hat{k}$ as $|a| + |b| + |c|$. This definition coincides with the usual definition of length of a vector $a\hat{i} + b\hat{j} + c\hat{k}$ if and only if

(1) $a = b = c = 0$
(2) any two of a, b and c are zero
(3) any one of a, b and c is zero
(4) $a + b + c = 0$

15. Given three vectors $\vec{a} = 6\hat{i} - 3\hat{j}$, $\vec{b} = 2\hat{i} - 6\hat{j}$ and $\vec{c} = -2\hat{i} + 21\hat{j}$ such that $\vec{\alpha} = \vec{a} + \vec{b} + \vec{c}$. Then the resolution of the vector $\vec{\alpha}$ into components with respect to $\vec{a}$ and $\vec{b}$ is given by

(1) $3\vec{a} - 2\vec{b}$ (2) $3\vec{b} - 2\vec{a}$ (3) $2\vec{a} - 3\vec{b}$ (4) $\vec{a} - 2\vec{b}$

16. If $\vec{\alpha} + \vec{\beta} + \vec{\gamma} = a\vec{\delta}$ and $\vec{\beta} + \vec{\gamma} + \vec{\delta} = b\vec{\alpha}$, $\vec{\alpha}$ and $\vec{\delta}$ are non-collinear, then $\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta}$ equals

(1) $a\vec{\alpha}$ (2) $b\vec{\delta}$ (3) 0 (4) $(a + b)\vec{\gamma}$

17. In triangle ABC, $\angle A = 30°$, H is the orthocentre and D is the midpoint of BC. Segment HD is produced to T such that $HD = DT$. The length AT is equal to

(1) $2BC$ (2) $3BC$ (3) $\frac{4}{3}BC$ (4) none of these

18. Let $\vec{r_1}, \vec{r_2}, \vec{r_3}, \cdots, \vec{r_n}$ be the position vectors of points $P_1, P_2, P_3, \cdots, P_n$ relative to the origin O. If the vector equation $a_1\vec{r_1} + a_2\vec{r_2} + \cdots + a_n\vec{r_n} = 0$ holds, then a similar equation will also hold w.r.t. to any other origin provided

(1) $a_1 + a_2 + \cdots + a_n = n$
(2) $a_1 + a_2 + \cdots + a_n = 1$
(3) $a_1 + a_2 + \cdots + a_n = 0$
(4) $a_1 = a_2 = a_3 = \cdots = a_n = 0$

19. Given three non-zero, non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$. $\vec{r_1} = p\vec{a} + q\vec{b} + \vec{c}$ and $\vec{r_2} = \vec{a} + p\vec{b} + q\vec{c}$. If the vectors $\vec{r_1} + 2\vec{r_2}$ and $2\vec{r_1} + \vec{r_2}$ are collinear, then (p, q) is

(1) $(0, 0)$ (2) $(1, -1)$ (3) $(-1, 1)$ (4) $(1, 1)$

20. If the vectors $\vec{a}$ and $\vec{b}$ are linearly independent satisfying $(\sqrt{3}\tan\theta + 1)\vec{a} + (\sqrt{3}\sec\theta - 2)\vec{b} = 0$, then the most general values of θ are

(1) $n\pi - \frac{\pi}{6}, n \in Z$ (2) $2n\pi \pm \frac{11\pi}{6}, n \in Z$ (3) $n\pi \pm \frac{\pi}{6}, n \in Z$ (4) $2n\pi + \frac{11\pi}{6}, n \in Z$

21. In a trapezium, vector $\overrightarrow{BC} = \alpha\overrightarrow{AD}$. We will then find that $\vec{p} = \overrightarrow{AC} + \overrightarrow{BD}$ is collinear with $\overrightarrow{AD}$. If $\vec{p} = \mu\overrightarrow{AD}$, then which of the following is true?

(1) $\mu = \alpha + 2$ (2) $\mu + \alpha = 1$ (3) $\alpha = \mu + 1$ (4) $\mu = \alpha + 1$

22. Vectors $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 4\hat{k}$ are so placed that the end point of one vector is the starting point of the next vector. Then the vectors are

(1) not coplanar
(2) coplanar but cannot form a triangle
(3) coplanar and form a triangle
(4) coplanar and can form a right-angled triangle

23. Vectors $\vec{a} = -4\hat{i} + 3\hat{k}$; $\vec{b} = 14\hat{i} + 2\hat{j} - 5\hat{k}$ are laid off from one point. Vector $\vec{d}$, which is being laid off from the same point dividing the angle between vectors $\vec{a}$ and $\vec{b}$ in equal halves and having the magnitude $\sqrt{6}$, is

(1) $\hat{i} + \hat{j} + 2\hat{k}$ (2) $\hat{i} - \hat{j} + 2\hat{k}$ (3) $\hat{i} + \hat{j} - 2\hat{k}$ (4) $2\hat{i} - \hat{j} - 2\hat{k}$

24. If $\hat{i} - 3\hat{j} + 5\hat{k}$ bisects the angle between $\hat{a}$ and $-\hat{i} + 2\hat{j} + 2\hat{k}$, where $\hat{a}$ is a unit vector, then

(1) $\hat{a} = \frac{1}{105}(41\hat{i} + 88\hat{j} - 40\hat{k})$ (2) $\hat{a} = \frac{1}{105}(41\hat{i} + 88\hat{j} + 40\hat{k})$

(3) $\hat{a} = \frac{1}{105}(-41\hat{i} + 88\hat{j} - 40\hat{k})$ (4) $\hat{a} = \frac{1}{105}(41\hat{i} - 88\hat{j} - 40\hat{k})$

25. If $4\hat{i} + 7\hat{j} + 8\hat{k}$, $2\hat{i} + 3\hat{j} + 4\hat{k}$ and $2\hat{i} + 5\hat{j} + 7\hat{k}$ are the position vectors of the vertices A, B and C, respectively, of triangle ABC, then the position vector of the point where the bisector of angle A meets BC is

(1) $\frac{2}{3}(-6\hat{i} - 8\hat{j} - 6\hat{k})$ (2) $\frac{2}{3}(6\hat{i} + 8\hat{j} + 6\hat{k})$ (3) $\frac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$ (4) $\frac{1}{3}(5\hat{j} + 12\hat{k})$

26. If $\vec{b}$ is a vector whose initial point divides the join of $5\hat{i}$ and $5\hat{j}$ in the ratio $k : 1$ and whose terminal point is the origin and $|\vec{b}| \le \sqrt{37}$, then k lies in the interval

(1) $[-6, -1/6]$
(2) $(-\infty, -6] \cup [-1/6, \infty)$
(3) $[0, 6]$
(4) none of these

27. Find the value of λ so that the points P, Q, R and S on the sides OA, OB, OC and AB, respectively, of a regular tetrahedron $OABC$ are coplanar. It is given that $\frac{OP}{OA}=\frac{1}{3}, \frac{OQ}{OB}=\frac{1}{2}, \frac{OR}{OC}=\frac{1}{3}$ and $\frac{OS}{AB}=\lambda$.

(1) $\lambda=\frac{1}{2}$ (2) $\lambda=-1$ (3) $\lambda=0$ (4) for no value of λ

28. 'I' is the incentre of triangle ABC whose corresponding sides are a, b, c, respectively. $a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}$ is always equal to

(1) $\vec{0}$ (2) $(a+b+c)\overrightarrow{BC}$ (3) $(\vec{a}+\vec{b}+\vec{c})\overrightarrow{AC}$ (4) $(a+b+c)\overrightarrow{AB}$

29. Let $x^2+3y^2=3$ be the equation of an ellipse in the x–y plane. A and B are two points whose position vectors are $-\sqrt{3}\hat{i}$ and $-\sqrt{3}\hat{i}+2\hat{k}$. Then the position vector of a point P on the ellipse such that $\angle APB=\pi/4$ is

(1) $\pm\hat{j}$ (2) $\pm(\hat{i}+\hat{j})$ (3) $\pm\hat{i}$ (4) none of these

30. Locus of the point P, for which $\overrightarrow{OP}$ represents a vector with direction cosine $\cos\alpha=\frac{1}{2}$ (where O is the origin) is

(1) a circle parallel to the y–z plane with centre on the x-axis
(2) a cone concentric with the positive x-axis having vertex at the origin and the slant height equal to the magnitude of the vector
(3) a ray emanating from the origin and making an angle of 60° with the x-axis
(4) a disc parallel to the y–z plane with centre on the x-axis and radius equal to $|\overrightarrow{OP}|\sin 60°$

31. If $\vec{x}$ and $\vec{y}$ are two non-collinear vectors and ABC is a triangle with side lengths a, b and c satisfying $(20a-15b)\vec{x}+(15b-12c)\vec{y}+(12c-20a)(\vec{x}\times\vec{y})=\vec{0}$, then triangle ABC is

(1) an acute-angled triangle (2) an obtuse-angled triangle
(3) a right-angled triangle (4) an isosceles triangle

32. A uni-modular tangent vector on the curve $x=t^2+2$, $y=4t-5$, $z=2t^2-6t$ at $t=2$ is

(1) $\frac{1}{3}(2\hat{i}+2\hat{j}+\hat{k})$ (2) $\frac{1}{3}(\hat{i}-\hat{j}-\hat{k})$ (3) $\frac{1}{6}(2\hat{i}+\hat{j}+\hat{k})$ (4) $\frac{2}{3}(\hat{i}+\hat{j}+\hat{k})$

33. If $\vec{x}$ and $\vec{y}$ are two non-collinear vectors and a, b and c represent the sides of a ΔABC satisfying $(a-b)\vec{x}+(b-c)\vec{y}+(c-a)(\vec{x}\times\vec{y})=0$, then ΔABC is (where $\vec{x}\times\vec{y}$ is perpendicular to the plane of $\vec{x}$ and $\vec{y}$)

(1) an acute-angled triangle (2) an obtuse-angled triangle
(3) a right-angled triangle (4) a scalene triangle

34. $\vec{A}$ is a vector with direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. Assuming the y–z plane as a mirror, the direction cosines of the reflected image of $\vec{A}$ in the y–z plane are

(1) $\cos\alpha, \cos\beta, \cos\gamma$ (2) $\cos\alpha, -\cos\beta, \cos\gamma$
(3) $-\cos\alpha, \cos\beta, \cos\gamma$ (4) $-\cos\alpha, -\cos\beta, -\cos\gamma$

Multiple Correct Answers Type

1. The vectors $x\hat{i}+(x+1)\hat{j}+(x+2)\hat{k}, (x+3)\hat{i}+(x+4)\hat{j}+(x+5)\hat{k}$ and $(x+6)\hat{i}+(x+7)\hat{j}+(x+8)\hat{k}$ are coplanar if x is equal to

(1) 1 (2) -3 (3) 4 (4) 0

2. The sides of a parallelogram are $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$. The unit vector parallel to one of the diagonals is

(1) $\frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$ (2) $\frac{1}{7}(3\hat{i}-6\hat{j}-2\hat{k})$

(3) $\frac{1}{\sqrt{69}}(\hat{i}+2\hat{j}+8\hat{k})$ (4) $\frac{1}{\sqrt{69}}(-\hat{i}-2\hat{j}+8\hat{k})$

3. A vector $\vec{a}$ has the components $2p$ and 1 w.r.t. a rectangular Cartesian system. This system is rotated through a certain angle about the origin in the counterclockwise sense. If, with respect to a new system, $\vec{a}$ has components $(p+1)$ and 1, then p is equal to

(1) -1 (2) $-1/3$ (3) 1 (4) 2

4. If points $\hat{i}+\hat{j}, \hat{i}-\hat{j}$ and $p\hat{i}+q\hat{j}+r\hat{k}$ are collinear, then

(1) $p=1$ (2) $r=0$ (3) $q \in R$ (4) $q \neq 1$

5. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\vec{a}+2\vec{b}+3\vec{c}, \lambda\vec{b}+\mu\vec{c}$ and $(2\lambda-1)\vec{c}$ are coplanar when

(1) $\mu \in R$ (2) $\lambda=\frac{1}{2}$ (3) $\lambda=0$ (4) no value of λ

6. If the resultant of three forces $\vec{F_1}=p\hat{i}+3\hat{j}-\hat{k}, \vec{F_2}=6\hat{i}-\hat{k}$ and $\vec{F_3}=-5\hat{i}+\hat{j}+2\hat{k}$ acting on a particle has a magnitude equal to 5 units, then the value of p is

(1) -6 (2) -4 (3) 2 (4) 4

7. If the vectors $\hat{i}-\hat{j}, \hat{j}+\hat{k}$ and $\vec{a}$ form a triangle, then $\vec{a}$ may be

(1) $-\hat{i}-\hat{k}$ (2) $\hat{i}-2\hat{j}-\hat{k}$ (3) $2\hat{i}+\hat{j}+\hat{k}$ (4) $\hat{i}+\hat{k}$

8. The vector $\hat{i}+x\hat{j}+3\hat{k}$ is rotated through an angle θ and is doubled in magnitude. It now becomes $4\hat{i}+(4x-2)\hat{j}+2\hat{k}$. The values of x are

(1) 1 (2) $-2/3$ (3) 2 (4) 4/3

9. $\vec{a}, \vec{b}$ and $\vec{c}$ are three coplanar unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$. If three vectors $\vec{p}, \vec{q}$ and $\vec{r}$ are parallel to $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, and have integral but different magnitudes, then among the following options, $|\vec{p}+\vec{q}+\vec{r}|$ can take a value equal to

(1) 1 (2) 0 (3) $\sqrt{3}$ (4) 2

10. If non-zero vectors $\vec{a}$ and $\vec{b}$ are equally inclined to coplanar vector $\vec{c}$, then $\vec{c}$ can be

(1) $\frac{|\vec{a}|}{|\vec{a}|+2|\vec{b}|}\vec{a}+\frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{b}$

(2) $\frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{a}+\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}\vec{b}$

(3) $\frac{|\vec{a}|}{|\vec{a}|+2|\vec{b}|}\vec{a}+\frac{|\vec{b}|}{|\vec{a}|+2|\vec{b}|}\vec{b}$

(4) $\frac{|\vec{b}|}{2|\vec{a}|+|\vec{b}|}\vec{a}+\frac{|\vec{a}|}{2|\vec{a}|+|\vec{b}|}\vec{b}$

11. If $A(-4, 0, 3)$ and $B(14, 2, -5)$, then which one of the following points lie on the bisector of the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (O is the origin of reference)?

(1) (2, 2, 4) (2) (2, 11, 5) (3) (–3, –3, –6) (4) (1, 1, 2)

12. In a four-dimensional space where unit vectors along the axes are $\hat{i}, \hat{j}, \hat{k}$ and $\hat{l}$, and $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are four non-zero vectors such that no vector can be expressed as a linear combination of others and $(\lambda-1)(\vec{a}_1-\vec{a}_2)+\mu(\vec{a}_2+\vec{a}_3)+\gamma(\vec{a}_3+\vec{a}_4-2\vec{a}_2)+\vec{a}_3+\delta\vec{a}_4=\vec{0}$, then

(1) $\lambda=1$ (2) $\mu=-2/3$ (3) $\gamma=2/3$ (4) $\delta=1/3$

13. Let ABC be a triangle, the position vectors of whose vertices are $7\hat{j}+10\hat{k}$, $-\hat{i}+6\hat{j}+6\hat{k}$ and $-4\hat{i}+9\hat{j}+6\hat{k}$. Then ΔABC is

(1) isosceles (2) equilateral (3) right angled (4) none of these

Linked Comprehension Type

For Problems 1–3

$ABCD$ is a parallelogram. L is a point on BC which divides BC in the ratio 1 : 2. AL intersects BD at P. M is a point on DC which divides DC in the ratio 1 : 2 and AM intersects BD in Q.

1. Point P divides AL in the ratio

(1) 1 : 2 (2) 1 : 3 (3) 3 : 1 (4) 2 : 1

2. Point Q divides DB in the ratio

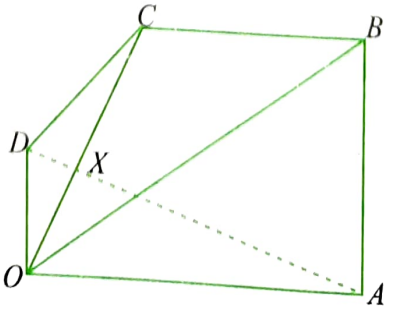
(1) 1 : 2 (2) 1: 3 (3) 3 : 1 (4) 2 : 1

3. $PQ : DB$ is equal to

(1) 2/3 (2) 1/3 (3) 1/2 (4) 3/4

For Problems 4 and 5

Let $OABCD$ be a pentagon in which the sides OA and CB are parallel and the sides OD and AB are parallel. Also $OA : CB = 2 : 1$ and $OD : AB = 1 : 3$.



4. The ratio $\frac{OX}{XC}$ is

(1) 3/4 (2) 1/3 (3) 2/5 (4) 1/2

5. The ratio $\frac{AX}{XD}$ is

(1) 5/2 (2) 6 (3) 7/3 (4) 4

For Problems 6 and 7

Consider the regular hexagon $ABCDEF$ with centre at O (origin).

6. $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC}$ is equal to

(1) $2\overrightarrow{AB}$ (2) $3\overrightarrow{AB}$ (3) $4\overrightarrow{AB}$ (4) none of these

7. Five forces $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{AE}, \overrightarrow{AF}$ act at the vertex A of a regular hexagon $ABCDEF$. Then their resultant is

(1) $3\overrightarrow{AO}$ (2) $2\overrightarrow{AO}$ (3) $4\overrightarrow{AO}$ (4) $6\overrightarrow{AO}$

For Problems 8 and 9

Let A, B, C, D, E represent vertices of a regular pentagon $ABCDE$. Given the position vector of these vertices be $\vec{a}, \vec{a}+\vec{b}, \vec{b}, \lambda\vec{a}$ and $\lambda\vec{b}$, respectively.

8. The ratio $\frac{AD}{BC}$ is equal to

(1) $1-\cos\frac{3\pi}{5} : \cos\frac{3\pi}{5}$ (2) $1+2\cos\frac{2\pi}{5} : \cos\frac{\pi}{5}$

(3) $1+2\cos\frac{\pi}{5} : 2\cos\frac{\pi}{5}$ (4) None of these

9. AD divides EC in the ratio

(1) $\cos\frac{2\pi}{5} : 1$ (2) $\cos\frac{3\pi}{5} : 1$ (3) $1 : 2\cos\frac{2\pi}{5}$ (4) $1 : 2$

For Problems 10 and 11

In a parallelogram $OABC$, vectors $\vec{a}, \vec{b}, \vec{c}$ are, respectively, the position vectors of vertices A, B, C with reference to O as origin. A point E is taken on the side BC which divides it in the ratio of 2 : 1. Also, the line segment AE intersects the line bisecting the angle $\angle AOC$ internally at point P. If CP when extended meets AB in point F, then

10. The position vector of point P is

(1) $\frac{|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{c}}{|\vec{c}|}\right)$ (2) $\frac{3|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{c}}{|\vec{c}|}\right)$

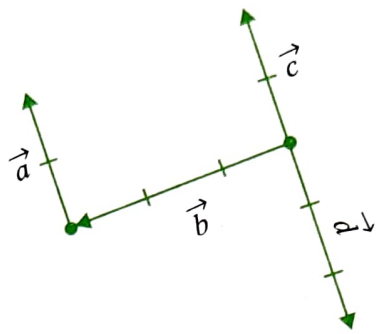
(3) $\frac{2|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{c}}{|\vec{c}|}\right)$ (4) None of these

11. The ratio in which F divides AB is

(1) $\frac{2|\vec{a}|}{\|\vec{a}|-3|\vec{c}\|}$ (2) $\frac{|\vec{a}|}{\|\vec{a}|-3|\vec{c}\|}$ (3) $\frac{3|\vec{a}|}{\|\vec{a}|-3|\vec{c}\|}$ (4) $\frac{3|\vec{c}|}{3|\vec{c}|-|\vec{a}|}$

Matrix Match Type

1. Refer to the following diagram and then match the lists:



List I	List II
a. Collinear vectors	**p.** $\vec{a}$
b. Coinitial vectors	**q.** $\vec{b}$
c. Equal vectors	**r.** $\vec{c}$
d. Unlike vectors (same initial point)	**s.** $\vec{d}$

2. $\vec{a}$ and $\vec{b}$ form the consecutive sides of a regular hexagon $ABCDEF$. Now, match the following lists

List I	List II
a. If $\overrightarrow{CD} = x\vec{a} + y\vec{b}$, then	**p.** $x = -2$
b. If $\overrightarrow{CE} = x\vec{a} + y\vec{b}$, then	**q.** $x = -1$
c. If $\overrightarrow{AE} = x\vec{a} + y\vec{b}$, then	**r.** $y = 1$
d. $\overrightarrow{AD} = -x\vec{b}$, then	**s.** $y = 2$

Numerical Value Type

1. Let ABC be a triangle whose centroid is G, orhtocentre is H and circumcentre is the origin 'O'. If D is any point in the plane of the triangle such that no three of O, A, C and D are collinear satisfying the relation $\vec{AD} + \vec{BD} + \vec{CH} + 3\vec{HG} = \lambda\vec{HD}$, then what is the value of the scalar 'λ'?

2. If the resultant of three forces $\vec{F_1} = p\hat{i} + 3\hat{j} - \hat{k}$, $\vec{F_2} = -5\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{F_3} = 6\hat{i} - \hat{k}$ acting on a particle has a magnitude equal to 5 units, then what is difference in the values of p?

3. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}+\vec{b}-\vec{c}=0$. If the area of triangle formed by vectors $\vec{a}$ and $\vec{b}$ is A, then what is the value of $4A^2$?
4. Find the least positive integral value of x for which the angle between vectors $\vec{a}=x\hat{i}-3\hat{j}-\hat{k}$ and $\vec{b}=2x\hat{i}+x\hat{j}-\hat{k}$ is acute.
5. Vectors along the adjacent sides of parallelogram are $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}+\hat{k}$. Find the length of the longer diagonal of the parallelogram.
6. If vectors $\vec{a}=\hat{i}+2\hat{j}-\hat{k}, \vec{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\lambda\hat{i}+\hat{j}+2\hat{k}$ are coplanar, then find the value of $(\lambda-4)$.

Archives

JEE Main

Single Correct Answer Type

1. If the vectors $\overrightarrow{AB}=3\hat{i}+4\hat{k}$ and $\overrightarrow{AC}=5\hat{i}-2\hat{j}+4\hat{k}$ are the sides of a triangle ABC, then the length of median through A is

(1) $\sqrt{18}$ (2) $\sqrt{72}$ (3) $\sqrt{33}$ (4) $\sqrt{45}$

(JEE Main 2013)

JEE Advanced

Matrix Match Type

1. Match the following lists:

List I	List II
a. In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z respectively. If $2(a^2-b^2)=c^2$ and $\lambda=\frac{\sin(X-Y)}{\sin Z}$ then possible values of n for which $\cos(n\pi\lambda)=0$ is (are)	**p.** 1
b. In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1+\cos 2X-2\cos 2Y=2\sin X\sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	**q.** 2
c. In R^2, let $\sqrt{3}\hat{i}+\hat{j}, \hat{i}+\sqrt{3}\hat{j}$ and $\beta\hat{i}+(1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect of the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	**r.** 3
d. Suppose that $F(\alpha)$ denotes the area of the region bounded by $x=0$, $x=2$, $y^2=4x$ and $y=\lvert\alpha x-1\rvert+\lvert\alpha x-2\rvert+\alpha x$, where $\alpha\in\{0,1\}$. Then the value(s) of $F(\alpha)+\frac{8}{3}\sqrt{2}$, when $\alpha=0$ and $\alpha=1$, is (are)	**s.** 5
	t. 6

(JEE Advanced 2015)

Numerical Value Type

1. Consider the set of eight vectors $V = \{a\hat{i} + b\hat{j} + c\hat{k} ; a, b, c \in \{-1, 1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is _____. **(JEE Advanced 2013)**

2. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p} + \vec{q} + \vec{r})$, $(\vec{p} - \vec{q} + \vec{r})$ and $(-\vec{p} - \vec{q} + \vec{r})$ are x, y and z, respectively, then the value of $2x + y + z$ is **(JEE Advanced 2015)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	**2.** (4)	**3.** (3)	**4.** (3)	**5.** (3)	**6.** (1)	**7.** (3)	**8.** (3)	**9.** (2)	**10.** (4)
11. (2)	**12.** (4)	**13.** (2)	**14.** (2)	**15.** (3)	**16.** (3)	**17.** (1)	**18.** (3)	**19.** (4)	**20.** (4)
21. (4)	**22.** (2)	**23.** (1)	**24.** (4)	**25.** (3)	**26.** (2)	**27.** (2)	**28.** (1)	**29.** (1)	**30.** (2)
31. (3)	**32.** (1)	**33.** (1)	**34.** (3)						

Multiple Correct Answers Type

1. (1), (2), (3), (4)	**2.** (1), (4)	**3.** (2), (3)	**4.** (1), (2), (4)	**5.** (1), (2), (3)
6. (2), (3)	**7.** (1), (2), (4)	**8.** (2), (3)	**9.** (3), (4)	**10.** (2), (4)
11. (1), (3), (4)	**12.** (1), (2), (4)	**13.** (1), (3)		

Linked Comprehension Type

1. (3)	**2.** (2)	**3.** (3)	**4.** (3)	**5.** (2)	**6.** (3)	**7.** (4)	**8.** (3)	**9.** (3)	**10.** (2)
11. (4)									

Matrix Match Type

1. a → p, r, s; b → q, r, s; c → p, r; d → r, s

2. a → q, r; b → p, r; c → q, s; d → p

Numerical Value Type

1. (2)	**2.** (6)	**3.** (3)	**4.** (2)	**5.** (7)	**6.** (9)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3)

JEE Advanced

Matrix Match Type

1. c → p, q

Numerical Value Type

1. (5) **2.** (9)

3 Different Products of Vectors

DOT (SCALAR) PRODUCT OF VECTORS

The scalar product of vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a} \cdot \vec{b}$, is defined to be the number $|\vec{a}||\vec{b}| \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, i.e.,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta, \text{ where } 0 \le \theta \le \pi.$$

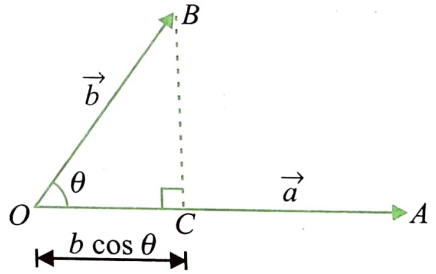
Notes:

- $\vec{a} \cdot \vec{b}$ is positive if θ is acute.
- $\vec{a} \cdot \vec{b}$ is negative if θ is obtuse.
- $\vec{a} \cdot \vec{b}$ is zero if θ is a right angle.

PHYSICAL INTERPRETATION OF SCALAR PRODUCT

Let $\overrightarrow{OA} = \vec{a}$ represent a force acting on a particle at O and let $\overrightarrow{OB} = \vec{b}$ represent the displacement of the particle from O to B as shown in the figure. Then the displacement in the direction of the force is $OC = b\cos\theta$.

Therefore, the work done by a force is a scalar quantity equal to the product of the magnitude of the force and the resolved part of the displacement in the direction of force. Work done by force $\vec{a}$ in moving its point of application from O to B is $|\vec{a}||\vec{b}|\cos\theta = \vec{a} \cdot \vec{b}$.

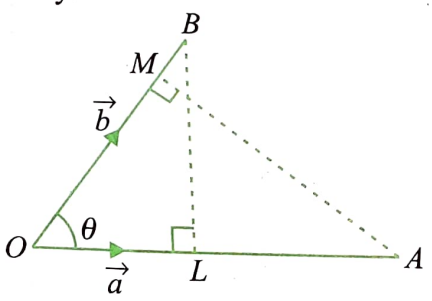


GEOMETRICAL INTERPRETATION OF SCALAR PRODUCT

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively.

Here OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$, respectively.

Now,
$$\begin{aligned}\vec{a} \cdot \vec{b} &= |\vec{a}||\vec{b}| \cos\theta \\ &= |\vec{a}|(OB\cos\theta) \\ &= |\vec{a}|(OL) \\ &= (\text{magnitude of } \vec{a}) \text{ (projection of } \vec{b} \text{ on } \vec{a})\end{aligned}$$



Again, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{b}|(|\vec{a}|\cos\theta)$

$= |\vec{b}|(OA\cos\theta)$

$= |\vec{b}|(OM)$

$=$ (magnitude of $\vec{b}$) (projection of $\vec{a}$ on $\vec{b}$)

Thus, geometrically interpreted, the scalar product of two vectors is the product of modulus of either vectors and the projection of the other in its direction. Thus,

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\frac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$$

$$\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$$

PROPERTIES OF DOT (SCALAR) PRODUCT

I. $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0° = |\vec{a}|^2 = a^2 \Rightarrow \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

II. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

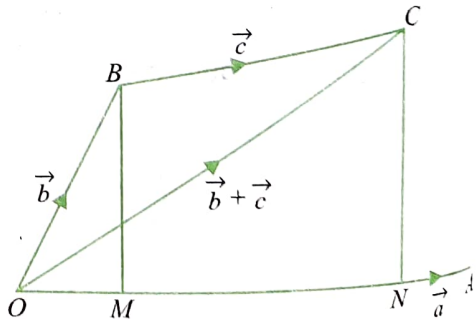
III. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)

Proof:

Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}, \vec{BC} = \vec{c}$ so that

$$\vec{OC} = \vec{OB} + \vec{BC} = \vec{b} + \vec{c}$$

From B draw $BM \perp OA$ and from C, drawn $CN \perp OA$



$$\text{L.H.S.} = \vec{a}\cdot(\vec{b}+\vec{c})$$
$$= \vec{OA}\cdot\vec{OC}$$
$$= (OA)(OC)\cos\theta \text{ (where } \theta = \angle CON)$$
$$= (OA)(ON) \text{ (as } ON = OC\cos\theta)$$
$$= (OA)(OM + MN)$$
$$= (OA)(OM) + (OA)(MN)$$
$$= \vec{OA}\cdot\vec{OB} + \vec{OA}\cdot\vec{BC}$$
$$= \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = \text{R.H.S.}$$

IV. $(l\vec{a})\cdot(m\vec{b}) = lm(\vec{a}\cdot\vec{b})$, where l and m are scalars

V. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are perpendicular to each other

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

VI. $(\vec{a} \pm \vec{b})^2 = (\vec{a} \pm \vec{b}) \cdot (\vec{a} \pm \vec{b})$

$= |\vec{a}|^2 + |\vec{b}|^2 \pm 2\vec{a} \cdot \vec{b}\, b$

$= |\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta$

VII. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

VIII. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$(\because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0)$

IX. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Taking dot product with $\hat{i}, \hat{j}$ and $\hat{k}$, alternatively, we have

$x = \vec{r} \cdot \hat{i},\ y = \vec{r} \cdot \hat{j}$ and $z = \vec{r} \cdot \hat{k}$

$\Rightarrow \quad \vec{r} = (\vec{r} \cdot \hat{i})\hat{i} + (\vec{r} \cdot \hat{j})\hat{j} + (\vec{r} \cdot \hat{k})\hat{k}$

APPLICATIONS OF DOT (SCALAR) PRODUCT

Finding Angle between Two Vectors

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are non-zero vectors, then the angle between them is given by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Also

$$\frac{(a_1b_1 + a_2b_2 + a_3b_3)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} = \cos^2\theta \le 1$$

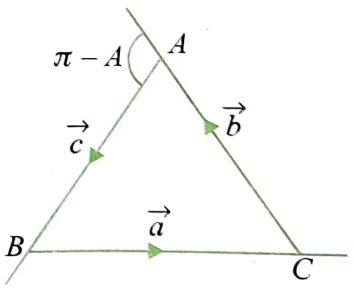
or $\quad (a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

Cosine Rule Using Dot Product

Using vector method, prove that in a triangle $a^2 = b^2 + c^2 - 2bc\cos A$ (cosine law)

In ΔABC,

Let $\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}$,



Since $\vec{a} + \vec{b} + \vec{c} = 0$, we have $\vec{a} = -(\vec{b} + \vec{c})$

$\therefore \quad |\vec{a}| = |-(\vec{b} + \vec{c})|$

or $\quad |\vec{a}|^2 = |\vec{b} + \vec{c}|^2$

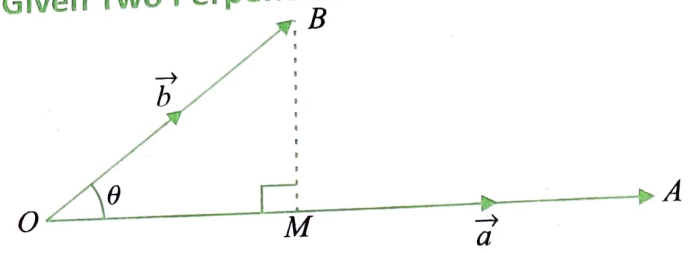
or $\quad |\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b} \cdot \vec{c}$

or $\quad |\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos(\pi - A)$

($\because$ angle between $\vec{b}$ and $\vec{c}$ = angle between CA produced and AB)

or $\quad a^2 = b^2 + c^2 - 2bc\cos A$

Finding Components of a Vector $\vec{b}$ Along and Perpendicular to Vector $\vec{a}$ or Resolving a Given Vector in the Direction of Given Two Perpendicular Vectors



Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively and let θ be the angle between $\vec{a}$ and $\vec{b}$. Then

$$\vec{b} = \overrightarrow{OM} + \overrightarrow{MB}$$

Also
$$\overrightarrow{OM} \quad (OM)\hat{a}$$
$$= (OB\cos\theta)\,\hat{a}$$
$$= (|\vec{b}|\cos\theta)\hat{a}$$
$$= \left(|\vec{b}|\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}||\vec{b}|}\right)\hat{a}$$
$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\hat{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{a}|}\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Also
$$\vec{b} = \overrightarrow{OM} + \overrightarrow{MB}$$

$\Rightarrow$
$$\overrightarrow{MB} = \vec{b} - \overrightarrow{OM} = \vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Thus, the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$, respectively.

ILLUSTRATION 3.1

Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

Sol. The given vectors are

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$
$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$

Now,
$$\vec{a}\cdot\vec{b} = (\hat{i} - 2\hat{j} + 3\hat{k})\cdot(3\hat{i} - 2\hat{j} + \hat{k})$$
$$= 1\cdot 3 + (-2)(-2) + 3\cdot 1$$
$$= 3 + 4 + 3$$
$$= 10$$

Now $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta.$

$\therefore$ $10=\sqrt{14}\sqrt{14}\cos\theta$

or $\cos\theta=\dfrac{10}{14}$

or $\theta=\cos^{-1}\left(\dfrac{5}{7}\right)$

ILLUSTRATION 3.2

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$, then find the geometrical relation between the vectors.

Sol. $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$

or $\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=\vec{0}$

or $\vec{a}\cdot(\vec{b}-\vec{c})=\vec{0}$

$\Rightarrow$ Either $\vec{b}-\vec{c}=\vec{0}$ or $\vec{a}\perp(\vec{b}-\vec{c})$

$\Rightarrow$ $\vec{b}=\vec{c}$ or $\vec{a}\ (\vec{b}-\vec{c})$

ILLUSTRATION 3.3

If $\vec{r}\cdot\hat{i}=\vec{r}\cdot\hat{j}=\vec{r}\cdot\hat{k}$ and $|\vec{r}|=3$, then find vector $\vec{r}$.

Sol. Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$. Since $\vec{r}\cdot\hat{i}=\vec{r}\cdot\hat{j}=\vec{r}\cdot\hat{k}$,

$x=y=z$

Also $|\vec{r}|=\sqrt{x^2+y^2+z^2}=3$

$\Rightarrow$ $x=\sqrt{3}$

Hence, the required vector $\vec{r}=\pm\sqrt{3}(\hat{i}+\hat{j}+\hat{k})$

ILLUSTRATION 3.4

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

Sol. Quaring $(\vec{a}+\vec{b}+\vec{c})=\vec{0}$, we get

$$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}=0$$

or $2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-3$

or $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\dfrac{3}{2}$

ILLUSTRATION 3.5

If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, then find the angle between vectors $\vec{a}$ and $\vec{a}+\vec{b}+\vec{c}$.

Sol. Since $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular, $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$

Angle between $\vec{a}$ and $\vec{a}+\vec{b}+\vec{c}$ is

$$\cos\theta=\frac{\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$$

Now $|\vec{a}|=|\vec{b}|=|\vec{c}|=a$

$$|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$$

$$=a^2+a^2+a^2+0+0+0$$

$$=3a^2$$

or $|\vec{a}+\vec{b}+\vec{c}|=\sqrt{3}a$

Putting this value in (i), we get $\theta=\cos^{-1}\frac{1}{\sqrt{3}}$

ILLUSTRATION 3.6

If $|\vec{a}|+|\vec{b}|=|\vec{c}|$ and $\vec{a}+\vec{b}=\vec{c}$, then find the angle between $\vec{a}$ and $\vec{b}$.

Sol. $\vec{a}+\vec{b}=\vec{c}$

or $|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{c}|^2$ (i)

and $|\vec{a}|+|\vec{b}|=|\vec{c}|$

or $|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|=|\vec{c}|^2$ (ii)

$\therefore$ $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$

$\Rightarrow$ $\cos\theta=1$ or $\theta=0^\circ$ [from (i) and (ii)]

ILLUSTRATION 3.7

If three unit vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the angle between $\vec{a}$ and $\vec{b}$.

Sol. $\vec{a}+\vec{b}=-\vec{c}$

or $|\vec{a}+\vec{b}|^2=|\vec{c}|^2=1$

$$|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = 1$$

or

$$\vec{a}\cdot\vec{b} = -\frac{1}{2}$$

or

$$|\vec{a}||\vec{b}|\cos\theta = -\frac{1}{2}$$

$\Rightarrow$

$$\cos\theta = \frac{1}{2}$$

or

$$\theta = \frac{2\pi}{3}$$

or

ILLUSTRATION 3.8

If θ is the angle between the unit vectors $\vec{a}$ and $\vec{b}$, then prove that

i. $\cos\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$

ii. $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}-\vec{b}|$

Sol.

i. $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

$$= 1 + 1 + 2(1)(1)\cos\theta$$

$$= 2 + 2\cos\theta$$

$\Rightarrow$ $|\vec{a}+\vec{b}|^2 = 2\cdot 2\cos^2\frac{\theta}{2}$

or $\cos\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$

ii. $(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$

$$= 1 + 1 - 2(1)(1)\cos\theta$$

$$= 2 - 2\cos\theta$$

$\Rightarrow$ $|\vec{a}-\vec{b}|^2 = 2\cdot 2\sin^2\frac{\theta}{2}$

or $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$

ILLUSTRATION 3.9

Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.

Sol. Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b} = 7\hat{i}-\hat{j}+8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})=\frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{1(7)+3(-1)+7(8)\}$$
$$=\frac{7-3+56}{\sqrt{49+1+64}}=\frac{60}{\sqrt{114}}$$

ILLUSTRATION 3.10

If the scalar projection of vector $x\hat{i}-\hat{j}+\hat{k}$ on vector $2\hat{i}-\hat{j}+5\hat{k}$ is $\frac{1}{\sqrt{30}}$, then find the value of x.

Sol. Projection of $x\hat{i}-\hat{j}+\hat{k}$ on $2\hat{i}-\hat{j}+5\hat{k}$

$$=\frac{(x\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+5\hat{k})}{\sqrt{4+1+25}}$$
$$=\frac{2x+1+5}{\sqrt{30}}$$

But, given $\frac{2x+6}{\sqrt{30}}=\frac{1}{\sqrt{30}}$ $\Rightarrow 2x+6=1$ or $x=\frac{-5}{2}$

ILLUSTRATION 3.11

If $\vec{a}=x\hat{i}+(x-1)\hat{j}+\hat{k}$ and $\vec{b}=(x+1)\hat{i}+\hat{j}+a\hat{k}$ make an acute angle $\forall\, x\in R$, then find the values of a.

Sol. $\vec{a}\cdot\vec{b}=(x\hat{i}+(x-1)\hat{j}+\hat{k})\cdot((x+1)\hat{i}+\hat{j}+a\hat{k})$

$$=x(x+1)+x-1+a$$
$$=x^2+2x+a-1$$

We must have $\vec{a}\cdot\vec{b}>0\ \forall\, x\in R$

$\Rightarrow$ $x^2+2x+a-1>0\ \forall\, x\in R$

$\Rightarrow$ $4-4(a-1)<0$

or $a>2$

ILLUSTRATION 3.12

If $\vec{a}\cdot\hat{i}=\vec{a}\cdot(\hat{i}+\hat{j})=\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})$, then find the unit vector $\vec{a}$.

Sol. Let $\vec{a}=x\hat{i}+y\hat{j}+z\hat{k}$

Then $\vec{a}\cdot\hat{i}=(x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{i}=x$ and $\vec{a}\cdot(\hat{i}+\hat{j})=x+y$

and $\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})=x+y+z$ (given that $x=x+y=x+y+z$)

Now $x=x+y\Rightarrow y=0$ and $x+y=x+y+z\Rightarrow z=0$

Hence, $x = 1$ (since $\vec{a}$ is a unit vector)

$\therefore \quad \vec{a} = \hat{i}$

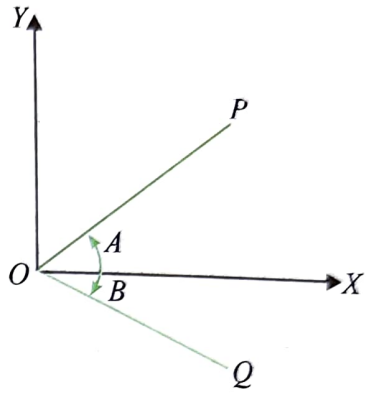
ILLUSTRATION 3.13

Prove by vector method that $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

Sol. Let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY, respectively.

Let $\vec{OP}, \vec{OQ}$ be two unit vectors drawn in the plane XOY such that $\angle XOP = A$, $\angle XOQ = B$

$\therefore \quad POQ = A + B$



Now $\quad \vec{OP} = \hat{i}\cos A + \hat{j}\sin A$

$\vec{OQ} = \hat{i}\cos B - \hat{j}\sin B$

$\therefore \quad \vec{OP} \cdot \vec{OQ} = \cos A \cos B - \sin A \sin B$

$\Rightarrow \quad (1)(1)\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\Rightarrow \quad \cos(A + B) = \cos A \cos B - \sin A \sin B$

ILLUSTRATION 3.14

In any triangle ABC, prove the projection formula $a = b\cos C + c\cos B$ using vector method.

Sol. Let $\vec{BC} = \vec{a}, \vec{CA} = \vec{b}, \vec{AB} = \vec{c}$, so that

$BC = a, \; CA = b, \; AB = c$

Now $\quad \vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\therefore \quad \vec{a} \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$

$\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 0$

$a^2 + ab\cos(180° - C) + ac\cos(180° - B) = 0$

$a^2 - ab\cos C - ac\cos B = 0$

$a - b\cos C - c\cos B = 0$

$a = b\cos C + c\cos B$

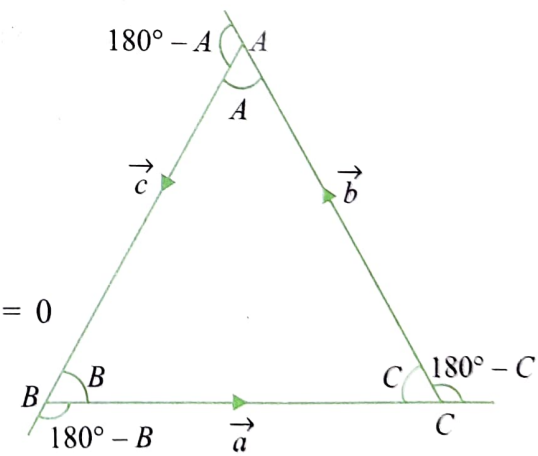
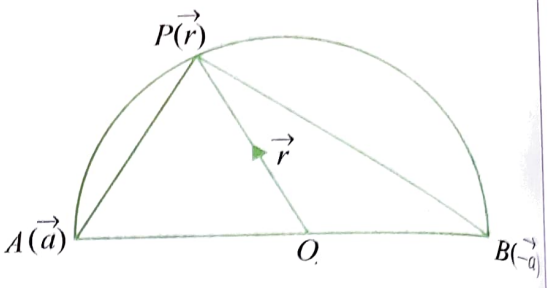


ILLUSTRATION 3.15

Prove that an angle inscribed in a semi-circle is a right angle using vector method.

Sol. Let O be the centre of the semi-circle and BA be the diameter. Let P be any point on the circumference of the semi-circle.

Let $\vec{OA} = \vec{a}$, then $\vec{OB} = -\vec{a}$



Let $\vec{OP} = \vec{r}$

$\therefore$ $\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}$

$\vec{BP} = \vec{OP} - \vec{OB} = \vec{r} - (-\vec{a}) = \vec{r} + \vec{a}$

$\vec{AP} \cdot \vec{BP} = (\vec{r} - \vec{a}) \cdot (\vec{r} + \vec{a})$

$= \vec{r}^2 - \vec{a}^2$

$= a^2 - a^2$ [$\because r = a$ as $OP = OA$]

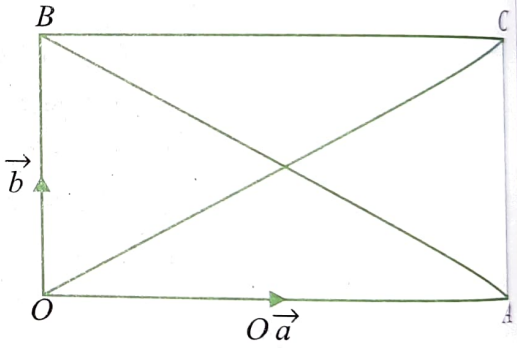
Therefore, $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BP}$.

$\Rightarrow$ $\angle APB = 90°$

ILLUSTRATION 3.16

Using dot product of vectors, prove that a parallelogram, whose diagonals are equal, is a rectangle.

Sol. Let $OACB$ be a parallelogram such that $OC = AB$.



Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$

Now $OC = AB$

or $OC^2 = AB^2$

or $\left(\vec{OA} + \vec{AC}\right)^2 = \left(\vec{AO} + \vec{OB}\right)^2$

or $\left(\vec{OA} + \vec{OB}\right)^2 = \left(-\vec{OA} + \vec{OB}\right)^2$

$\Rightarrow$ $(\vec{a} + \vec{b}) = (-\vec{a} + \vec{b})^2$

or $\vec{a}^2 + \vec{b}^2 + 2\vec{a} \cdot \vec{b} = \vec{a}^2 + \vec{b}^2 - 2\vec{a} \cdot \vec{b}$

or $2\vec{a} \cdot \vec{b} = -2\vec{a} \cdot \vec{b}$

or $4\vec{a} \cdot \vec{b} = 0$

or $\vec{a} \cdot \vec{b} = \vec{0}$

Hence, $\vec{a}$ and $\vec{b}$ are perpendicular, i.e.,

$\angle AOB = 90°$

Therefore, $OACB$ is a rectangle.

ILLUSTRATION 3.17

If $a + 2b + 3c = 4$, then find the least value of $a^2 + b^2 + c^2$.

Sol. Consider vectors $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{q} = \hat{i} + 2\hat{j} + 3\hat{k}$

Now $$\cos\theta = \frac{a + 2b + 3c}{\sqrt{a^2 + b^2 + c^2}\sqrt{1^2 + 2^2 + 3^2}}$$

or $$\cos^2\theta = \frac{(a + 2b + 3c)^2}{14(a^2 + b^2 + c^2)} \le 1$$

$\Rightarrow$ $$a^2 + b^2 + c^2 \ge \frac{8}{7}$$

Hence, least value of $a^2 + b^2 + c^2$ is $\frac{8}{7}$.

ILLUSTRATION 3.18

Find a unit vector $\vec{a}$ which makes an angle of $\pi/4$ with the z-axis and it is such that $(\vec{a} + \hat{i} + \hat{j})$ is a unit vector.

Sol. Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Given $|\vec{a}| = 1$, therefore

$$x^2 + y^2 + z^2 = 1 \qquad \text{(i)}$$

Angle between $\vec{a}$ and z-axis is $\pi/4$; therefore,

$$\cos\left(\frac{\pi}{4}\right) = \frac{\vec{a}\cdot\hat{k}}{|\vec{a}||\hat{k}|}$$

$\Rightarrow$ $$z = \frac{1}{\sqrt{2}}$$

Now $$\vec{a} + \hat{i} + \hat{j} = (x+1)\hat{i} + (y+1)\hat{j} + z\hat{k}$$

Given that $\vec{a} + \hat{i} + \hat{j}$ is a unit vector. Therefore,

$$|\vec{a} + \hat{i} + \hat{j}| = \sqrt{[(x+1)^2 + (y+1)^2 + z^2]} = 1$$

or $$x^2 + y^2 + z^2 + 2x + 2y + 1 = 0$$

or $$1 + 2x + 2y + 1 = 0 \qquad \text{[using (i)]}$$

or $$y = -(x + 1)$$

From (i), we have

$$x^2 + (x + 1)^2 + (1/2) = 1$$

$\Rightarrow$ $$4x^2 + 4x + 1 = 0 \text{ or } (2x + 1)^2 = 0$$

$$x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$$

Hence, $$\vec{a} = -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

ILLUSTRATION 3.19

Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are of the same length and when taken pair-wise they form equal angles. If $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$, then find vector $\vec{c}$.

Sol. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$. Then $|\vec{a}| = |\vec{b}| = |\vec{c}| \Rightarrow x^2 + y^2 + z^2 = 2$

It is given that the angles between the vectors taken in pairs are equal, say θ. Therefore,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{0+1+0}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} = \frac{1}{2} \text{ and } \frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{x+y}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \text{ and } \frac{y+z}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow \quad x + y = 1 \text{ and } y + z = 1$$

$$\Rightarrow \quad y = 1 - x \text{ and } z = 1 - y = 1 - (1 - x) = x$$

Also $\quad x^2 + y^2 + z^2 = 2 \Rightarrow x^2 + (1-x)^2 + x^2 = 2$

$$\Rightarrow \quad (3x + 1)(x - 1) = 0 \Rightarrow x = 1, -1/3$$

Now, $\quad y = 1 - x \Rightarrow y = 0$ for $x = 1$ and $y = 4/3$ for $x = -1/3$

Hence, $\quad \vec{c} = \hat{i} + 0\hat{j} + \hat{k}$ and $\vec{c} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} - \frac{1}{3}\hat{k}$

ILLUSTRATION 3.20

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors and $\vec{d}$ is a unit vector which makes equal angles with $\vec{a}, \vec{b}$ and $\vec{c}$, then find the value of $|\vec{a} + \vec{b} + \vec{c} + \vec{d}|^2$.

Sol. $|\vec{a} + \vec{b} + \vec{c} + \vec{d}|^2 = \Sigma|\vec{a}|^2 + 2\Sigma\,\vec{a}\cdot\vec{b} = 4 + 2\vec{d}\cdot(\vec{a} + \vec{b} + \vec{c})$ $\quad (\because \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular).

Let $\vec{d} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$. Then $\vec{d}\cdot\vec{a} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = \cos\theta$. Therefore,

$$\lambda = \mu = \nu = \cos\theta$$

Also $\quad \lambda^2 + \mu^2 + \nu^2 = 1 \Rightarrow 3\cos^2\theta = 1$ or $\cos\theta = \frac{1}{\sqrt{3}}$

$$\therefore \quad |\vec{a} + \vec{b} + \vec{c} + \vec{d}|^2 = 4 \pm \frac{2\cdot 3}{\sqrt{3}} = 4 \pm 2\sqrt{3}$$

ILLUSTRATION 3.21

A particle acted by constant forces $4\hat{i}+\hat{j}-3\hat{k}$ and $3\hat{i}+\hat{j}-\hat{k}$ is displaced from point $\hat{i}+2\hat{j}+3\hat{k}$ to point $5\hat{i}+4\hat{j}+\hat{k}$. Find the total work done by the forces in units.

Sol. Here $\vec{F}=\vec{F_1}+\vec{F_2}=(4\hat{i}+\hat{j}-3\hat{k})+(3\hat{i}+\hat{j}-\hat{k})=7\hat{i}+2\hat{j}-4\hat{k}$

and $\vec{d}=\vec{d_2}-\vec{d_1}\ (5\hat{i}+4\hat{j}+\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=4\hat{i}+2\hat{j}-2\hat{k}$

$\therefore$ Work done $=\vec{F}\cdot\vec{d}$

$$=(7\hat{i}+2\hat{j}-4\hat{k})\cdot(4\hat{i}+2\hat{j}-2\hat{k})$$
$$=(7)(4)+(2)(2)+(-4)(-2)$$
$$=28+4+8=40 \text{ units}$$

ILLUSTRATION 3.22

If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$.

Sol. Since $\vec{a},\vec{b}$, and $\vec{c}$ are mutually perpendicular vectors, we have $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$.

It is given that

$$|\vec{a}|=|\vec{b}|=|\vec{c}|$$

Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b}$ and $\vec{c}$ at angles θ_1, θ_2 and θ_3, respectively. Then, we have

$$\cos\theta_1=\frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$
$$=\frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}[\vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{a}=0]$$
$$=\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_2=\frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|}$$
$$=\frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|}[\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{b}=0]$$
$$=\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_3 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|} = \frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{c}|}$$

$$= \frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{c}|} [\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c} = 0]$$

$$= \frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Now as $|\vec{a}| = |\vec{b}| = |\vec{c}|$, $\cos\theta_1 = \cos\theta_2 = \cos\theta_3$

$\therefore \quad \theta_1 = \theta_2 = \theta_3$

Hence, the vector $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$.

ILLUSTRATION 3.23

If $\vec{a} = 4\hat{i}+6\hat{j}$ and $\vec{b} = 3\hat{j}+4\hat{k}$, then find the component of $\vec{a}$ along $\vec{b}$.

Sol. The component of vector $\vec{a}$ along $\vec{b}$ is $\dfrac{(\vec{a}.\vec{b})\vec{b}}{|\vec{b}|^2} = \dfrac{18}{25}(3\hat{i}+4\hat{k})$

ILLUSTRATION 3.24

If $|\vec{a}| = |\vec{b}| = |\vec{a}+\vec{b}| = 1$, then find the value of $|\vec{a}-\vec{b}|$.

Sol. We have

$$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$$

or $\quad 1+|\vec{a}-\vec{b}|^2 = 4 \quad$ or $\quad |\vec{a}-\vec{b}| \sqrt{3}$.

ILLUSTRATION 3.25

If $\vec{a} = -\hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = 2\hat{i}+0\hat{j}+\hat{k}$, then find vector $\vec{c}$ satisfying the following conditions: **(i)** that it is coplanar with $\vec{a}$ and $\vec{b}$, **(ii)** that it is $\perp$ to $\vec{b}$ and **(iii)** that $\vec{a}\cdot\vec{c} = 7$.

Sol. Let $\vec{c} = x\hat{i}+y\hat{j}+z\hat{k}$

Then from condition (i),

$$\begin{vmatrix} x & y & z \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0 \text{ or } x+3y-2z = 0 \quad \text{(i)}$$

From condition (ii),

$$2x+z = 0$$

(ii)

From condition (iii),

$$-x+y+z=7 \qquad \text{(iii)}$$

Solving (i), (ii) and (iii), we get the values of x, y and z; hence, vector $\vec{c}=\frac{1}{2}(-3\hat{i}+5\hat{j}+6\hat{k})$

ILLUSTRATION 3.26

Let $\vec{a}, \vec{b}$ and $\vec{c}$ are vectors such that $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{c}|=5$, and $(\vec{a}+\vec{b})$ is perpendicular to $\vec{c}$, $(\vec{b}+\vec{c})$ is perpendicular to $\vec{a}$ and $(\vec{c}+\vec{a})$ is perpendicular to $\vec{b}$. Then find the value of $|\vec{a}+\vec{b}+\vec{c}|$.

Sol. Given, $(\vec{a}+\vec{b})\cdot\vec{c}=0 \Rightarrow \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}=0$

$$(\vec{b}+\vec{c})\cdot\vec{a}=0 \Rightarrow \vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{a}=0$$

$$(\vec{c}+\vec{a})\cdot\vec{b}=0 \Rightarrow \vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{b}=0$$

$\therefore$ $$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$$

Now, $$|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=50$$

$\Rightarrow$ $$|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$$

ILLUSTRATION 3.27

Prove that in a tetrahedron if two pairs of opposite edges are perpendicular, then the third pair is also perpendicular.

Sol. Let $ABCD$ be the tetrahedron and A be at the origin.

Let $$\overrightarrow{AB}=\vec{b},\ \overrightarrow{AC}=\vec{c} \text{ and } \overrightarrow{AD}=\vec{d}$$

Let the edge AB be perpendicular to the opposite edge CD. Then

$$\overrightarrow{AB}\cdot\overrightarrow{CD}=0$$

or $$\vec{b}\cdot(\vec{d}-\vec{c})=0$$

or $$\vec{b}\cdot\vec{d}=\vec{b}\cdot\vec{c} \qquad \text{(i)}$$

Also let AC be perpendicular to the opposite edge BD. Therefore,

$$\overrightarrow{AC}\cdot\overrightarrow{BD}=0$$

or $$\vec{c}\cdot(\vec{d}-\vec{b})=0$$

or $$\vec{c}\cdot\vec{d}=\vec{b}\cdot\vec{c} \qquad \text{(ii)}$$

Now from (i) and (ii), we have

$$\vec{b}\cdot\vec{d}=\vec{c}\cdot\vec{d}$$

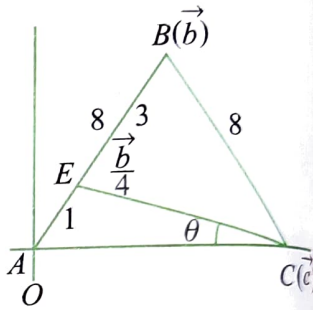
or $(\vec{c}-\vec{b})\cdot\vec{d}=0$

or $\overrightarrow{BC}\cdot\overrightarrow{AD}=0$

Hence, AD is perpendicular to opposite edge BC.

ILLUSTRATION 3.28

In isosceles triangle ABC, $|\overrightarrow{AB}|=|\overrightarrow{BC}|=8$, a point E divides AB internally in the ratio $1:3$, then find the angle between $\overrightarrow{CE}$ and $\overrightarrow{CA}$ (where $|\overrightarrow{CA}|=12$).



Sol. Given $|\vec{c}|=12$ and $|\vec{b}|=|\vec{b}-\vec{c}|=8$

$\Rightarrow$ $b^2=b^2+c^2-\vec{b}\cdot\vec{c}$

$\Rightarrow$ $\vec{b}\cdot\vec{c}=72$

$$\cos\theta\ \frac{\vec{c}.\left(\vec{c}-\frac{\vec{b}}{4}\right)}{|\vec{c}|\left|\vec{c}-\frac{\vec{b}}{4}\right|}=\frac{\vec{c}\cdot\vec{c}-\frac{\vec{c}.\vec{b}}{4}}{12\left|\vec{c}-\frac{\vec{b}}{4}\right|}=\frac{144-18}{12\left|\vec{c}-\frac{\vec{b}}{4}\right|}$$

Now $$\left|\vec{c}-\frac{\vec{b}}{4}\right|^2=|\vec{c}|^2+\frac{|\vec{b}|^2}{16}-\frac{\vec{b}\cdot\vec{c}}{2}=144+4-36=112$$

$\Rightarrow$ $$\cos\theta=\frac{21}{2\times\sqrt{112}}=\frac{21}{2\times4\sqrt{7}}=\frac{3\sqrt{7}}{8}$$

ILLUSTRATION 3.29

Arc AC of a circle subtends a right angle at the centre O. Point B divides the arc in the ratio $1:2$. If $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$, then calculate $\overrightarrow{OC}$ in terms of $\vec{a}$ and $\vec{b}$.

Sol. Vector $\vec{c}$ is coplanar with vector $\vec{a}$ and $\vec{b}$. Therefore,

$$\vec{c}=x\vec{a}+y\vec{b}$$

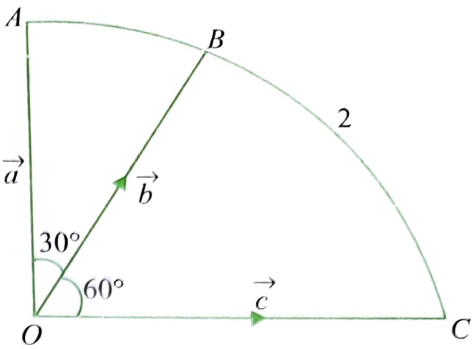
Point B divides arc AC in the ratio $1:2$ so that $\angle AOB=30°$ and $\angle BOC=60°$.

We have to find the values of x and y when we are given

$$|\vec{a}|=|\vec{b}|=|\vec{c}|=r\ (\text{say}).$$

$\vec{a}\cdot\vec{b} = r^2\cos 30° = r^2\frac{\sqrt{3}}{2}$ and $\vec{a}\cdot\vec{c} = 0$

$\vec{b}\cdot\vec{c} = r^2\cos 60° = \frac{r^2}{2}$



Multiplying both sides of (i) scalarly by $\vec{c}$ and $\vec{a}$, we get

$\vec{c}\cdot\vec{c} = x\,\vec{a}\cdot\vec{c} + y\,\vec{b}\cdot\vec{c}$

and $\vec{c}\cdot\vec{a} = x\,\vec{a}\cdot\vec{a} + y\,\vec{b}\cdot\vec{a}$

$r^2 = 0 + \frac{r^2}{2}y,\ y = 2$

and $0 = xr^2 + yr^2\frac{\sqrt{3}}{2}$

Putting $y = 2,\ x = -\sqrt{3}$

$\vec{c} = -\sqrt{3}\,\vec{a} + 2\,\vec{b}$

ILLUSTRATION 3.30

Vector $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 2\hat{k}$ turns through a right angle passing through the positive x-axis on the way. Show that the vector in its new position is $\frac{4\hat{i} - \hat{j} - \hat{k}}{\sqrt{2}}$.

Sol. Let the new vector be $\overrightarrow{OB} = x\hat{i} + y\hat{j} + z\hat{k}$.

According to the given condition, we have

$$|\overrightarrow{OB}| = |\overrightarrow{OA}| = 3 \Rightarrow x^2 + y^2 + z^2 = 9 \quad \text{(i)}$$

Also $\overrightarrow{OA}\ \overrightarrow{OB} \Rightarrow x + 2y + 2z = 0$ (ii)

Since while turning $\overrightarrow{OA}$, it passes through the positive x-axis on the way, vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\lambda\hat{i}$ are coplanar. Thus,

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 2 \\ \lambda & 0 & 0 \end{vmatrix} = 0$$

or $y - z = 0$ (iii)

Solving (i), (ii) and (iii) for x, y and z, we have $x = -4y = -4z$

$\Rightarrow$ $16y^2 + y^2 + y^2 = 9$

$\Rightarrow$ $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$ and $x = \mp\frac{1}{\sqrt{2}}$

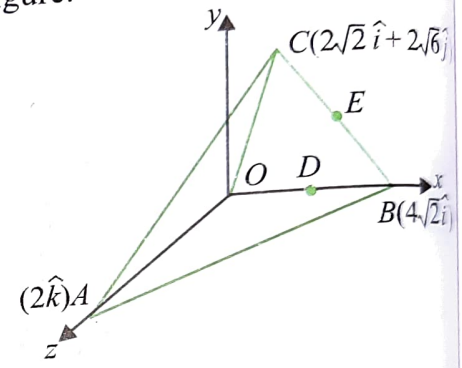
$$\Rightarrow \quad \overrightarrow{OB} = \pm\left(\frac{4}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}\right)$$

Since angle between $\overrightarrow{OB}$ and $\hat{i}$ is acute, $\overrightarrow{OB} = \frac{4}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}$

ILLUSTRATION 3.31

The base of the pyramid $AOBC$ is an equilateral triangle OBC with each side equal to $4\sqrt{2}$, O is the origin of reference, AO is perpendicular to the plane of ΔOBC and $|\overrightarrow{AO}| = 2$. Then find the cosine of the angle between the skew straight lines, one passing through A and the mid point of OB and the other passing through O and the mid point of BC.

Sol. According to the question, we have tetrahedron as shown in the figure.



$$\overrightarrow{AD} = 2\sqrt{2}\hat{i} - 2\hat{k}$$

and $$\overrightarrow{OE} = 3\sqrt{2}\hat{i} + \sqrt{6}\hat{j}$$

Cosine of angle between the vectors is

$$\cos\theta = \frac{12}{\sqrt{12}\sqrt{24}}$$

$$= \frac{1}{\sqrt{2}}$$

CONCEPT APPLICATION EXERCISE 3.1

1. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.
2. Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is a perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a}$ and $\vec{b}$.
3. If the vertices A, B, C of a triangle ABC are $(1, 2, 3)$, $(-1, 0, 0)$, $(0, 1, 2)$, respectively, then find $\angle ABC$.
4. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and the angle between $\vec{a}$ and $\vec{b}$ is $120°$, then find the value of $|4\vec{a} + 3\vec{b}|$.
5. If vectors $\hat{i} - 2x\hat{j} - 3y\hat{k}$ and $\hat{i} + 3x\hat{j} + 2y\hat{k}$ are orthogonal to each other, then find the locus of the point (x, y).
6. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be pairwise mutually perpendicular vectors, such that $|\vec{a}| = 1$, $|\vec{b}| = 2$, $|\vec{c}| = 2$. Then find the length of $\vec{a} + \vec{b} + \vec{c}$.
7. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$.
8. If the angle between unit vectors $\vec{a}$ and $\vec{b}$ is $60°$, then find the value of $|\vec{a} - \vec{b}|$.
9. Let $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{i} - \hat{j}$ and $\vec{w} = \hat{i} + 2\hat{j} + 3\hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{u}\cdot\hat{n} = 0$ and $\vec{v}\cdot\hat{n} = 0$, then find the value of $|\vec{w}\cdot\hat{n}|$.

10. A, B, C, D are any four points, prove that $\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{BC}\cdot\overrightarrow{AD}+\overrightarrow{CA}\cdot\overrightarrow{BD}=0$.

11. $P(1, 0, -1)$, $Q(2, 0, -3)$, $R(-1, 2, 0)$ and $S(3, -2, -1)$, then find the projection length of $\overrightarrow{PQ}$ on $\overrightarrow{RS}$.

12. If the vectors $3\vec{p}+\vec{q}$; $5\vec{p}-3\vec{q}$ and $2\vec{p}+\vec{q}$; $4\vec{p}-2\vec{q}$ are pairs of mutually perpendicular vectors, then find the angle between vectors $\vec{p}$ and $\vec{q}$.

13 Let $\vec{A}$ and $\vec{B}$ be two non-parallel unit vectors in a plane. If $(\alpha\vec{A}+\vec{B})$ bisects the internal angle between $\vec{A}$ and $\vec{B}$, then find the value of α.

14. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors, such that $\vec{a}+\vec{b}+\vec{c}=\vec{x}$, $\vec{a}\cdot\vec{x}=1$, $\vec{b}\cdot\vec{x}=\frac{3}{2}$, $|\vec{x}|=2$. Then find the angle between $c\,\vec{c}$ and $x\,\vec{x}$.

15. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the greatest value of $|\vec{a}+\vec{b}|+|\vec{a}-\vec{b}|$.

16. Constant forces $P_1=\hat{i}-\hat{j}+\hat{k}$, $P_2=-\hat{i}+2\hat{j}-\hat{k}$ and $P_3=\hat{j}-\hat{k}$ act on a particle at a point A. Determine the work done when particle is displaced from position $A(4\hat{i}-3\hat{j}-2\hat{k})$ to $B(6\hat{i}+\hat{j}-3\hat{k})$.

17. If $|\vec{a}|=5, |\vec{a}-\vec{b}|=8$ and $|\vec{a}+\vec{b}|=10$, then find $|\vec{b}|$.

18. If A, B, C, D are four distinct points in space such that $\overrightarrow{AB}$ is not perpendicular to $\overrightarrow{CD}$ and satisfies $\overrightarrow{AB}\cdot\overrightarrow{CD}=k(|\overrightarrow{AD}|^2+|\overrightarrow{BC}|^2-|\overrightarrow{AC}|^2-|\overrightarrow{BD}|^2)$, then find the value of k.

ANSWERS

1. $|\vec{a}|=\frac{16\sqrt{2}}{3\sqrt{7}}$, $|\vec{b}|=\sqrt{\frac{8}{63}}$

3. $\angle ABC=\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$

4. 12

5. $6x^2+6y^2=1$

6. $|\vec{a}+\vec{b}+\vec{c}|=3$

7. $\pi/3$

8. 1

9. $|\vec{w}\cdot\hat{n}|=3$

11. $\frac{6}{\sqrt{33}}$

12. $\cos^{-1}\frac{3}{8}$

13. 1

14. $\cos^{-1}(3/4)$

15. $2\sqrt{2}$

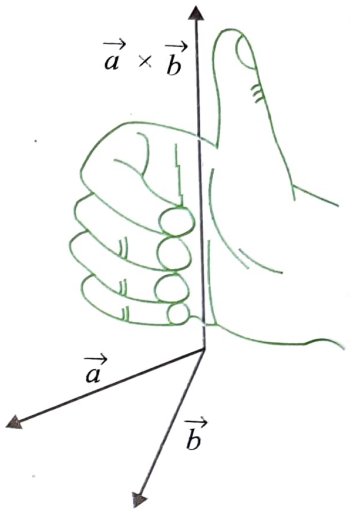
16. 9 units

17. $\sqrt{57}$

18. $k=1/2$

VECTOR (OR CROSS) PRODUCT OF TWO VECTORS

The cross product is just a shorthand invented for the purpose of quickly writing down the angular momentum of an object. Here's how the cross product arises naturally from angular momentum. Recall that if we have a fixed axis and an object at distance r away with velocity v and mass m is moving around the axis in a circle, then the magnitude of the angular momentum is $m|r||v|$, where $|r|$ is the magnitude of vector r. But what direction should the angular momentum vector point in? Well, if you follow the path of the object, it lies in a plane, an infinite two-dimensional surface. One way to represent a plane is to write down two different vectors that lie in the plane.



Another method used by mathematicians to represent a plane is to write down a single vector that is normal to the plane (normal is a synonym for perpendicular). If a plane is a flat sheet, the normal vector points straight up. Now, for any plane, there are two vectors that are normal to it, since if a vector n is normal to a plane, $-n$ will be normal as well. So how do we determine whether to use n or $-n$?

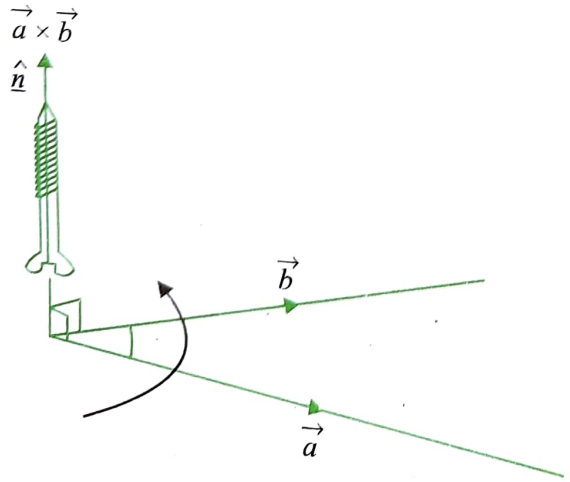
A long time ago, physicists made an arbitrary decision known today as the right-hand rule. Given vectors $\vec{a}$ and $\vec{b}$, just curl your fingers from $\vec{a}$ to $\vec{b}$ and the thumb points in the direction of the normal used.

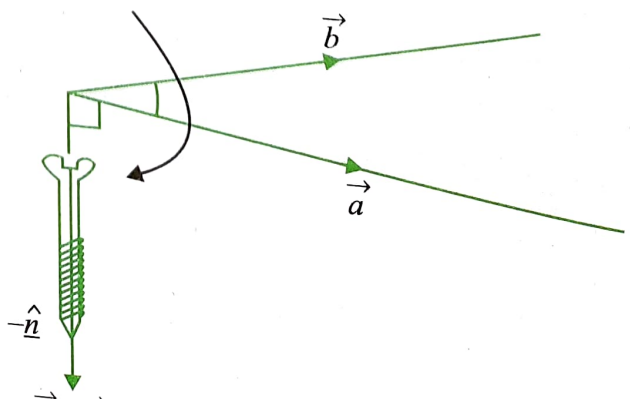
The vector product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a} \times \vec{b}$, is the vector $\vec{c} = |a||b| \sin\theta \hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, and $\hat{n}$ is a unit vector along the line perpendicular to both $\vec{a}$ and $\vec{b}$.

Then direction of $\vec{c}$ is such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right-handed system.

We see that the direction of $\vec{b} \times \vec{a}$ is opposite to that of $\vec{a} \times \vec{b}$ as shown in the figure.

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$$





So, the

PROPERTIES OF CROSS PRODUCT

I. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

II. $\vec{a} \times \vec{a} = 0$

III. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

IV. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

V. Two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if $\vec{a} \times \vec{b} = \vec{0}$.

VI. If $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\hat{i} + (a_3 b_1 - a_1 b_3)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

VII. The unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\dfrac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$, and a vector of magnitude λ perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm \dfrac{\lambda(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

PHYSICAL INTERPRETATION OF CROSS PRODUCT AS A MOMENT OF FORCE

Moment of force (often just *moment*) is the tendency of a force to twist or rotate an object. This is an important, basic concept in engineering and physics. A moment is valued mathematically as the product of the force and the moment arm. Moment arm is the perpendicular distance from the point of rotation to the *line of action* of the force. The moment may be thought of as a measure of the tendency of the force to cause rotation about an imaginary axis through a point.

Image A

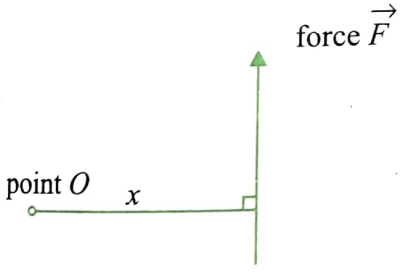


Image B

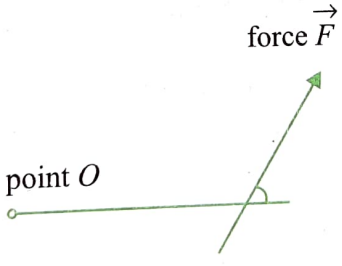
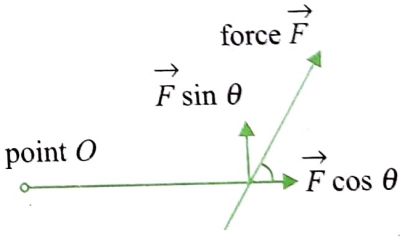


Image C



The moment of a force can be calculated about any point and not just the points in which the line of action of the force is perpendicular.

Image A shows the components, the force F and the moment arm x when they are perpendicular to one another. When the force is not perpendicular to the point of interest, such as point O in Images B and C, the magnitude of moment $\vec{M}$ of a vector $\vec{F}$ about point O is

$\vec{M}_O = \vec{r}_{OF} \times \vec{F}$, where $\vec{r}_{OF}$ is the vector from point O to the position where quantity F is applied.

Image C represents the vector components of the force in Image B. In order to determine moment $\vec{M}$ of vector $\vec{F}$ about point O, when vector $\vec{F}$ is not perpendicular to point O, one must resolve the force $\vec{F}$ into its horizontal and vertical components. The sum of the moments of the two components of F about point O is

$$\vec{M}_{OF} = \vec{F}\sin\theta(x) + \vec{F}\cos\theta(0)$$

The moment arm to the vertical component of $\vec{F}$ is a distance x. The moment arm to the horizontal component of $\vec{F}$ does not exist. There is no rotational force about point O due to the horizontal component of $\vec{F}$. Thus, the moment arm distance is zero.

Thus, $\vec{M}$ can be referred to as "moment $\vec{M}$ with respect to the axis that goes through point O", or simply "moment $\vec{M}$ about point O". If O is the origin, or informally, if the axis involved is clear from context, one often omits O and says simply *moment*, rather than *moment about O*. Therefore, the moment about *point O* is indeed the cross product, $\vec{M}_O = \vec{r}_{OF} \times \vec{F}$, since the cross product $= \vec{F}\sin\theta(x)$.

GEOMETRIC INTERPRETATION OF CROSS PRODUCT

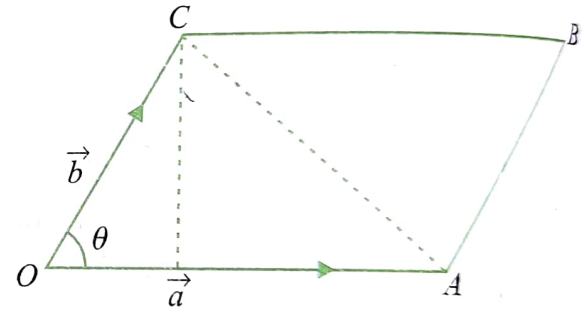
I. $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

or $$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$= 2\left(\frac{1}{2}|\vec{a}||\vec{b}|\sin\theta\right)$$

$$= 2\,(\text{Area of triangle } AOC)$$

$$= \text{Area of parallelogram}$$



Area of the triangle OAB is $\frac{1}{2}|\vec{a} \times \vec{b}|$.

$\vec{a} \times \vec{b}$ is said to be the vector area of the parallelogram with adjacent sides OA and OB.

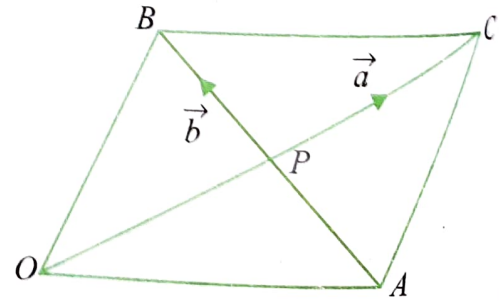
II. If $\vec{a}, \vec{b}$ are diagonals of a parallelogram, its area $= \frac{1}{2}|\vec{a} \times \vec{b}|$.

In the figure $\overrightarrow{OC} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$

$\Rightarrow$ Area parallelogram $= 4 \times \frac{1}{2}|\overrightarrow{PC} \times \overrightarrow{PB}|$

$$= 4 \times \frac{1}{2}\left|\frac{\vec{a}}{2} \times \frac{\vec{b}}{2}\right|$$

$$= \frac{1}{2}|\vec{a} \times \vec{b}|$$



III. If AC and BD are the diagonals of a quadrilateral, then its vector area is $\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$.

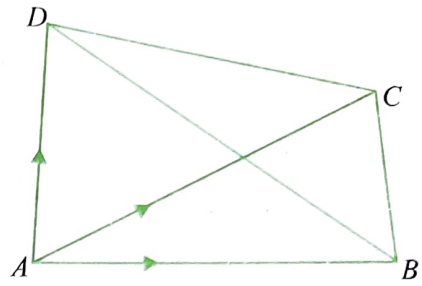
Vector area of the quadrilateral $ABCD$ = Vector area of ΔABC + Vector area of ΔACD

$$= \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC} + \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AD}$$

$$= -\frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{AD}$$

$$= \frac{1}{2}\overrightarrow{AC} \times (\overrightarrow{AD} - \overrightarrow{AB})$$

$$= \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$$



IV. The area of a triangle whose vertices are $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$.

$$\text{Area of triangle} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|$$

$$= \frac{1}{2}|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}|$$

$$= \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$$

ILLUSTRATION 3.32

Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$.

Sol. $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= \hat{i}(-14+14) - \hat{j}(2-21) + \hat{k}(-2+21) = 19\hat{j}+19\hat{k}$$

$\therefore$ $|\vec{a}\times\vec{b}| = \sqrt{(19)^2+(19)^2} = \sqrt{2\times(19)^2} = 19\sqrt{2}$

ILLUSTRATION 3.33

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is?

Sol. It is given that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$.

We know that $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}$ is a unit vector if $|\vec{a}\times\vec{b}| = 1$.

or $\quad ||a||b|\sin\ \hat{n}|\ \ 1$

or $\quad ||\vec{a}||\vec{b}|\sin\theta|=1$

or $\quad 3\times\dfrac{\sqrt{2}}{3}\times\sin\theta=1$

or $\quad \theta=\dfrac{\pi}{4}$

Hence, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{4}$.

ILLUSTRATION 3.34

Show that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$.

Sol. $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=(\vec{a}-\vec{b})\times\vec{a}+(\vec{a}-\vec{b})\times\vec{b}$ [By distributivity of vector product over addition]

$=\vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b}$ [Again, by distributivity of vector product over addition]

$=\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}$

$=2\vec{a}\times\vec{b}$

ILLUSTRATION 3.35

Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k},\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}\cdot\vec{d}=15$.

Sol. Vector $\vec{d}$ is perpendicular to vectors $\vec{a}=\hat{i}+4\hat{j}+2\hat{k},\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$

$\therefore\quad \vec{d}=\lambda\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 4 & 2\\ 3 & -2 & 7\end{vmatrix}=\lambda(32\hat{i}-\hat{j}-14\hat{k})$

Also $\vec{c}\cdot\vec{d}=15$

$\therefore\quad \lambda(2\hat{i}-\hat{j}+4\hat{k})\cdot(32\hat{i}-\hat{j}-14\hat{k})=15$

or $\quad 9\lambda=15$

or $\quad \lambda=\dfrac{5}{3}$

Hence, the required vector $\vec{d}=\dfrac{1}{3}(160\hat{i}-5\hat{j}-70\hat{k})$

ILLUSTRATION 3.36

If A, B and C are the vertices of a triangle ABC, then prove sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Sol. Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, $\vec{AB} = \vec{c}$, so that $\vec{a} + \vec{b} = -\vec{c}$. Therefore,

$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$$

$$\vec{0} + \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$$

$$|\vec{a} \times \vec{b}| = |\vec{c} \times \vec{a}|$$

$$ab \sin(180° - C) = ca \sin(180° - B)$$

$$ab \sin C = ca \sin B$$

Dividing both sides by abc, we get

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(i)}$$

Similarly, $\frac{c}{\sin C} = \frac{a}{\sin A}$ (ii)

From (i) and (ii), we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

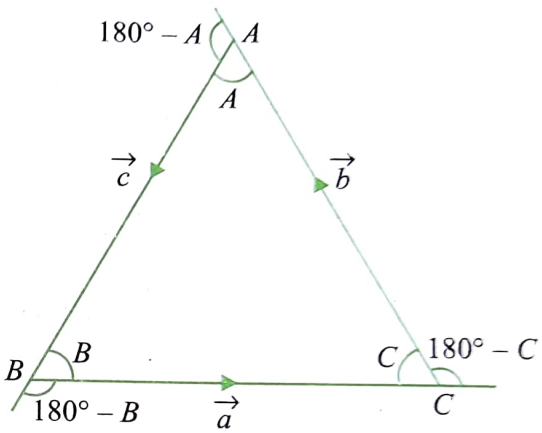


ILLUSTRATION 3.37

Using cross product of vectors, prove that $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

Sol. Let OP and OQ be unit vectors making angles A and B with the X-axis such that

$$\angle POQ = A + B$$

$$\therefore \quad \vec{OP} = \hat{i} \cos A + \hat{j} \sin A$$

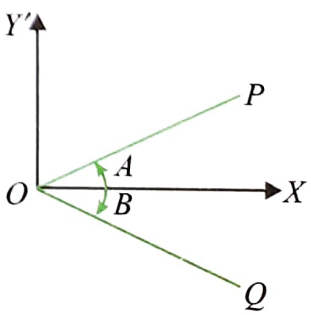
$$\vec{OQ} = \hat{i} \cos B - \hat{j} \sin B$$

Now $\vec{OP} \times \vec{OQ}$

$$= (1)(1) \sin(A + B)(-\hat{k})$$

$$= -\sin(A + B)\hat{k}$$

(i)



(ii)

Also

$$\vec{OP} \times \vec{OQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos A & \sin A & 0 \\ \cos B & -\sin B & 0 \end{vmatrix}$$

$$= (-\cos A \sin B - \sin A \cos B)\hat{k}$$

$$= -(\sin A \cos B + \cos A \sin B)\hat{k}$$

From (i) and (ii), we get

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

ILLUSTRATION 3.38

Find a unit vector perpendicular to the plane determined by the points (1, –1, 2), (2, 0, –1) and (0, 2, 1).

Sol. Given points are $A(1, -1, 2)$, $B(2, 0, -1)$ and $C(0, 2, 1)$

$$\Rightarrow \quad \overrightarrow{AB} = \vec{a} = \hat{i} + \hat{j} - 3\hat{k},\ \overrightarrow{BC} = \vec{b} = -2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -2 & 2 & 2 \end{vmatrix} = 8\hat{i} + 4\hat{j} + 4\hat{k}$$

Hence. Unit vector $= \pm \dfrac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$

ILLUSTRATION 3.39

If $\vec{a}$ and $\vec{b}$ are two vectors, then prove that $(\vec{a} \times \vec{b})^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix}$.

Sol.

$$(\vec{a} \times \vec{b})^2 = (ab \sin\theta \cdot \hat{n})^2$$
$$= a^2b^2 \sin^2\theta$$
$$= a^2b^2 - a^2b^2 \cos^2\theta$$
$$= (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$$
$$= \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}$$

ILLUSTRATION 3.40

If $|\vec{a}| = 2$, then find the value of $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2$.

Sol.

$$|\vec{a} \times \hat{i}|^2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix}^2 \qquad (\because \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$= |a_3\hat{j} - a_2\hat{k}|^2 = a_3^2 + a_2^2$$

Similarly, $|\vec{a}\times\vec{j}|^2=a_1^2+a_3^2$ and $|\vec{a}\times\vec{k}|^2=a_1^2+a_2^2$

Hence, the required result can be given as $2(a_1^2+a_2^2+a_3^2)=2|\vec{a}|^2=8$

ILLUSTRATION 3.41

$\vec{r}\times\vec{a}=\vec{b}\times\vec{a}$; $\vec{r}\times\vec{b}=\vec{a}\times\vec{b}$; $\vec{a}\neq\vec{0}$; $\vec{b}\neq\vec{0}$; $\vec{a}\neq\lambda\vec{b}$, and $\vec{a}$ is not perpendicular to $\vec{b}$, then find $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$.

Sol. $\vec{r}\times\vec{a}-\vec{b}\times\vec{a}=0$ and $\vec{r}\times\vec{b}+\vec{b}\times\vec{a}=0$

Adding, we get $\vec{r}\times(\vec{a}+\vec{b})=0$

But as we are given $\vec{a}\neq\lambda\vec{b}$, therefore

$$\vec{r}=\mu(\vec{a}+\vec{b})$$

ILLUSTRATION 3.42

A,B, C and D are any four points in the space, then prove that $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|=$ 4(area of ΔABC).

Sol. Let P.V. of A, B, C and D be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{0}$, respectively. Then

$$\overrightarrow{AB}\times\overrightarrow{CD}=(\vec{b}-\vec{a})\times(-\vec{c}),\ \overrightarrow{BC}\times\overrightarrow{AD}=(\vec{c}-\vec{b})\times(-\vec{a})$$

and $$\overrightarrow{CA}\times\overrightarrow{BD}=(\vec{a}-\vec{c})\times(-\vec{b})$$

$$\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}$$

$$=\vec{c}\times\vec{b}+\vec{a}\times\vec{c}+\vec{a}\times\vec{c}+\vec{b}\times\vec{a}-\vec{a}\times\vec{b}+\vec{c}\times\vec{b}$$

$$=2(\vec{c}\times\vec{b}+\vec{b}\times\vec{a}+\vec{a}\times\vec{c})$$

$$=2(\vec{c}\times(\vec{b}-\vec{a})-\vec{a}\times(\vec{b}-\vec{a}))$$

$$=2((\vec{c}-\vec{a})\times(\vec{b}-\vec{a}))$$

$$=2(\overrightarrow{AC}\times\overrightarrow{AB})$$

$\Rightarrow$ $$|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|=4\left|\frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AB})\right|=4\Delta ABC$$

ILLUSTRATION 3.43

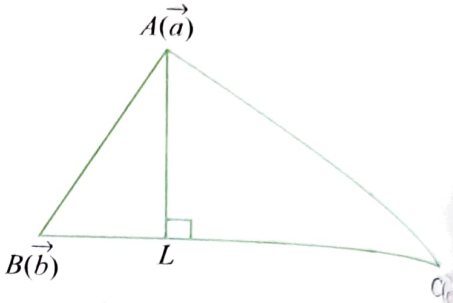
If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of the vertices A, B and C, respectively, of ΔABC, prove that the perpendicular distance of the vertex A from the base BC of the triangle ABC is $\dfrac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{c}-\vec{b}|}$.

Sol. $|\overrightarrow{BC} \times \overrightarrow{BA}| = |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$

or $|\overrightarrow{BC}||\overrightarrow{BA}| \sin B = |\vec{a} \times \vec{b} \times \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$

or $|\vec{c} - \vec{b}|(AB \sin B) = |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$

Therefore, the length of perpendicular from A on BC is



$$AL = AB \sin B = \frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{|\vec{b} - \vec{c}|}.$$

ILLUSTRATION 3.44

Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Sol. The vertices of triangle ABC are given as A (1, 1, 2), B (2, 3, 5), and C (1, 5, 5). The adjacent sides $\overline{AB}$ and $\overline{BC}$ of ΔABC are given as

$$\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BC}|$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \quad |\overrightarrow{AB} \times \overrightarrow{BC}| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36 + 9 + 16} = \sqrt{61}$$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

ILLUSTRATION 3.45

Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Sol. The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$. Adjacent sides are given as

$$\vec{a} = \hat{i} - \hat{j} + 3\hat{k} \text{ and } \vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$$

$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$|\vec{a}\times\vec{b}|=\sqrt{20^2+5^2+(-5)^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units.

ILLUSTRATION 3.46

Find the area of a parallelogram whose diagonals are $\vec{a}=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}-3\hat{j}+4\hat{k}$.

Sol. $$\Delta=\frac{1}{2}|\vec{a}\times\vec{b}|$$

But $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&1&-2\\1&-3&4\end{vmatrix}=-2\hat{i}-14\hat{j}-10\hat{k}$$

Hence, $$\Delta=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}\sqrt{4+196+100}=5\sqrt{3}$$

ILLUSTRATION 3.47

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a}\neq 0$, $|\vec{a}|=|\vec{c}|=1$, $|\vec{b}|=4$ and $|\vec{b}\times\vec{c}|=\sqrt{15}$. If $\vec{b}-2\vec{c}=\lambda\vec{a}$, then find the value of λ.

Sol. Let the angle between $\vec{b}$ and $\vec{c}$ be α. Then

$$|\vec{b}\times\vec{c}|=\sqrt{15}$$

or $$|\vec{b}||\vec{c}|\sin\alpha=\sqrt{15}$$

or $$\sin\alpha=\frac{\sqrt{15}}{4}$$

or $$\cos\alpha=\frac{1}{4}$$

$\Rightarrow$ $$\vec{b}-2\vec{c}=\lambda\vec{a}$$

or $$|\vec{b}-2\vec{c}|^2=\lambda^2|\vec{a}|^2$$

or $$|\vec{b}|^2+4|\vec{c}|^2-4\cdot\vec{b}\cdot\vec{c}=\lambda^2|\vec{a}|^2$$

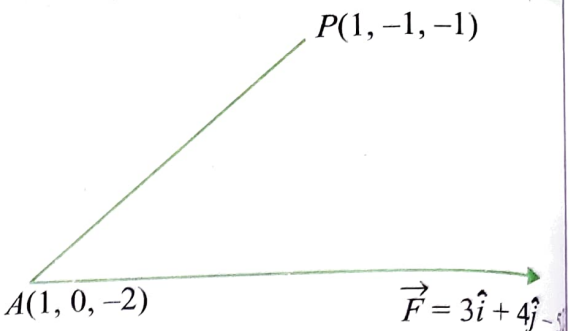
or $$16+4-4\{|\vec{b}||\vec{c}|\cos\alpha\}=\lambda^2$$

or $$16+4-4\times4\times1\times\frac{1}{4}=\lambda^2$$

or $$\lambda^2=16 \text{ or } \lambda=\pm4$$

ILLUSTRATION 3.48

Find the moment about $(1, -1, -1)$ of the force $3\hat{i} + 4\hat{j} - 5\hat{k}$ acting at $(1, 0, -2)$.



Sol.

$$\vec{F} = 3\hat{i} + 4\hat{j} - 5\hat{k}$$

$$\vec{PA} = \text{P.V. of A} - \text{P.V. of } P$$

$$= (\hat{i} - 2\hat{j}) - (\hat{i} - \hat{j} - \hat{k})$$

$$= -\hat{j} + \hat{k}$$

Required vector moment $= \vec{PA} \times \vec{F}$

$$= (-\hat{j} + \hat{k}) \times (3\hat{i} + 4\hat{j} - 5\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 3 & 4 & -5 \end{vmatrix}$$

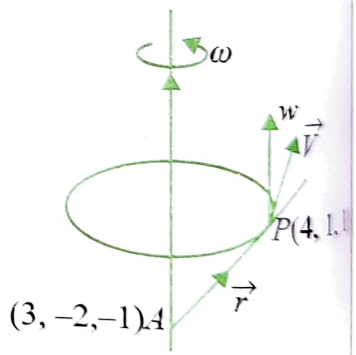
$$= \hat{i} + 3\hat{j} + 3\hat{k}$$

ILLUSTRATION 3.49

A rigid body is spinning about a fixed point $(3, -2, -1)$ with an angular velocity of 4 rad/s, the axis of rotation being in the direction of $(1, 2, -2)$. Find the velocity of the particle at point $(4, 1, 1)$.

Sol.

$$\vec{\omega} = 4\left(\frac{\hat{i} + 2\hat{j} - 2\hat{k}}{\sqrt{1+4+4}}\right) = \frac{4}{3}(\hat{i} + 2\hat{j} - 2\hat{k})$$



$$\vec{r} = \overrightarrow{OP} - \overrightarrow{OA}$$

$$= (4\hat{i} + \hat{j} + \hat{k}) - (3\hat{i} - 2\hat{j} - \hat{k})$$

$$= \hat{i} + 3\hat{j} + 2\hat{k}$$

$$\vec{v} = \vec{\omega} \times \vec{r} = \frac{4}{3}(\hat{i} + 2\hat{j} - 2\hat{k}) \times (\hat{i} + 3\hat{j} + 2\hat{k})$$

$$= \frac{4}{3}(10\hat{i} - 4\hat{j} + \hat{k})$$

ILLUSTRATION 3.50

If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then show that $\vec{a} - \vec{d}$, is parallel to $\vec{b} - \vec{c}$ provided $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

(i)

Sol. We have $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$

and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$

$\vec{a}-\vec{d}$ will be parallel to $\vec{b}-\vec{c}$

if $(\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=\vec{0}$

if $\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{d}\times\vec{b}+\vec{d}\times\vec{c}=\vec{0}$

i.e., if $(\vec{a}\times\vec{b}+\vec{d}\times\vec{c})-(\vec{a}\times\vec{c}+\vec{d}\times\vec{b})=\vec{0}$

i.e., if $(\vec{a}\times\vec{b}-\vec{c}\times\vec{d})-(\vec{a}\times\vec{c}-\vec{b}\times\vec{d})=\vec{0}$ [from (i)]

i.e., if $\vec{0}-\vec{0}=\vec{0}$

i.e., $\vec{0}=\vec{0}$, which is true

Hence, the result is true.

ILLUSTRATION 3.51

Show by a numerical example and geometrically also that $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ does not imply $\vec{b}=\vec{c}$.

Sol. Let $\vec{a}=3\hat{i}+2\hat{j}+5\hat{k}$, $\vec{b}=6\hat{i}+5\hat{j}+8\hat{k}$, $\vec{c}=3\hat{i}+3\hat{j}+3\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&2&5\\6&5&8\end{vmatrix}$$

$$=(16-25)\hat{i}-(24-30)\hat{j}+(15-12)\hat{k}$$

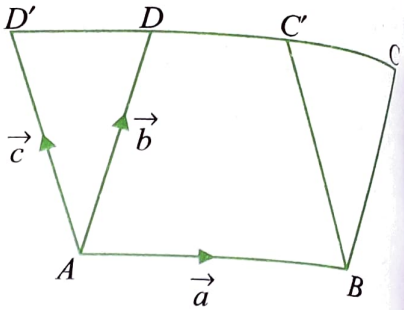
$$=-9\hat{i}+6\hat{j}+3\hat{k}$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&2&5\\3&3&3\end{vmatrix}$$

$$=(6-15)\hat{i}-(9-15)\hat{j}+(9-6)\hat{k}=-9\hat{i}+6\hat{j}+3\hat{k}$$

$\therefore$ $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, but $\vec{b}\neq\vec{c}$.

Geometrically



Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$, $\overrightarrow{AD'} = \vec{c}$. Then

Vector area of parallelogram $ABCD = \vec{a} \times \vec{b}$

Vector area of parallelogram $ABC'D' = \vec{a} \times \vec{c}$

Now Vector area of parallelogram $ABCD$
= Vector area of parallelogram $ABC'D'$
($\because$ both parallelograms have same base and same height)

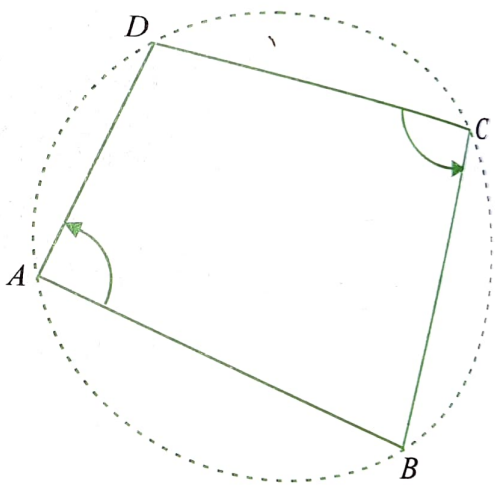
$\therefore$ $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ but $\vec{b} \neq \vec{c}$

ILLUSTRATION 3.52

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of the vertices of a cyclic quadrilateral $ABCD$, prove that

$$\frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a}|}{(\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{a})} + \frac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}{(\vec{b} - \vec{c}) \cdot (\vec{d} - \vec{c})} = 0.$$

Sol. Consider



$$\frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a}|}{(\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{a})} = \frac{|(\vec{a} - \vec{d}) \times (\vec{b} - \vec{a})|}{(\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{a})}$$

$$= \frac{|\vec{a} - \vec{d}|\,|\vec{b} - \vec{a}| \sin A}{|\vec{b} - \vec{a}|\,|\vec{d} - \vec{a}| \cos A}$$

$$= \tan A \qquad \text{(i)}$$

Also $$\frac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}{(\vec{b} - \vec{c}) \cdot (\vec{d} - \vec{c})} = \frac{|(\vec{b} - \vec{c}) \times (\vec{c} - \vec{d})|}{(\vec{b} - \vec{c}) \cdot (\vec{d} - \vec{c})}$$

$$= \frac{|\vec{b} - \vec{c}|\,|\vec{c} - \vec{d}| \sin C}{|\vec{b} - \vec{c}| \cdot |\vec{d} - \vec{c}| \cos C}$$

$$= \tan C$$

As it is a cyclic quadrilateral, we have (iii)

$$A = 180° - C$$

or $$\tan A = \tan (180° - C)$$

or $$\tan A + \tan C = 0$$

$\Rightarrow$ $$\frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a}|}{(\vec{b} - \vec{a}) \cdot (\vec{d} - \vec{a})} + \frac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}{(\vec{b} - \vec{c}) \cdot (\vec{d} - \vec{c})} = 0$$

ILLUSTRATION 3.53

The position vectors of the vertices of a quadrilateral with A as origin are $B(\vec{b})$, $D(\vec{d})$ and $C(l\vec{b}+m\vec{d})$. Prove that the area of the quadrilateral is $\frac{1}{2}(l+m)|\vec{b}\times\vec{d}|$.

Sol. Area of quadrilateral $= \frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}| = \frac{1}{2}|(l\vec{b}+m\vec{d})\times(\vec{d}-\vec{b})|$

$$= \frac{1}{2}|l\vec{b}\times\vec{d} - m\vec{d}\times\vec{b}|$$

$$= \frac{1}{2}(l+m)|\vec{b}\times\vec{d}|$$

ILLUSTRATION 3.54

Let $\vec{a}$ and $\vec{b}$ be unit vectors such that $|\vec{a}+\vec{b}|=\sqrt{3}$. Then find the value of $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b})$.

Sol. $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b}) = 6\vec{a}\cdot\vec{a}+17\vec{a}\cdot\vec{b}+5\vec{b}\cdot\vec{b} = 11+17\vec{a}\cdot\vec{b}$

$[\because \vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{a}\times\vec{b}) = 0$, as $\vec{a}$ and $\vec{b}$ are perpendicular to $\vec{a}\times\vec{b}]$

Now $|\vec{a}+\vec{b}| = \sqrt{3}$

or $|\vec{a}+\vec{b}|^2 = 3$

or $|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b} = 3$

or $\vec{a}\cdot\vec{b} = \frac{1}{2}$

$\Rightarrow$ $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b}) = 11+\frac{17}{2} = \frac{39}{2}$

ILLUSTRATION 3.55

$\hat{u}$ and $\hat{v}$ are two non-collinear unit vectors such that $\left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right| = 1$. Prove that $|\hat{u}\times\hat{v}| = \left|\frac{\hat{u}-\hat{v}}{2}\right|$

Sol. Given that $\left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right| = 1$

or $\left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right|^2 = 1$

$\Rightarrow$ $\frac{2+2\cos\theta}{4}+\sin^2\theta = 1$ $\quad(\because \hat{u}\cdot(\hat{u}\times\hat{v}) = \hat{v}\cdot(\hat{u}\times\hat{v}) = 0)$

$$\Rightarrow \quad \cos^2\frac{\theta}{2} = \cos^2\theta$$

$$\Rightarrow \quad \theta = n\pi \pm \frac{\theta}{2},\ n \in Z$$

or $$= \frac{2\pi}{3}$$

$$\Rightarrow \quad |\hat{u} \times \hat{v}| = \sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \left|\frac{\hat{u} - \hat{v}}{2}\right|$$

ILLUSTRATION 3.56

In triangle ABC, points D, E and F are taken on the sides BC, CA and AB, respectively, such that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = n$. Prove that $\Delta_{DEF} = \frac{n^2 - n + 1}{(n+1)^2}\Delta_{ABC}$.

Sol. Take A as the origin and let the position vectors of points B and C be $\vec{b}$ and $\vec{c}$, respectively.

Therefore, the position vectors of D, E and F are, respectively, $\frac{n\vec{c} + \vec{b}}{n+1}$, $\frac{\vec{c}}{n+1}$ and $\frac{n\vec{b}}{n+1}$. Therefore,

$$\overrightarrow{ED} = \overrightarrow{AD} - \overrightarrow{AE} = \frac{(n-1)\vec{c} + \vec{b}}{n+1} \text{ and } \overrightarrow{EF} = \frac{n\vec{b} - \vec{c}}{n+1}$$

Now $$\text{Vector area of } \Delta ABC = \frac{1}{2}(\vec{b} \times \vec{c})$$

and $$\text{Vector area of } \Delta DEF = \frac{1}{2}(\overrightarrow{EF} \times \overrightarrow{ED})$$

$$= \frac{1}{2(n+1)^2}[(n\vec{b} - \vec{c}) \times \{(n-1)\vec{c} + \vec{b}\}]$$

$$= \frac{1}{2(n+1)^2}[(n^2 - n)\,\vec{b} \times \vec{c} + \vec{b} \times \vec{c}]$$

$$= \frac{1}{2(n+1)^2}[(n^2 - n + 1)(\vec{b} \times \vec{c})] = \frac{n^2 - n + 1}{(n+1)^2}\Delta_{ABC}$$

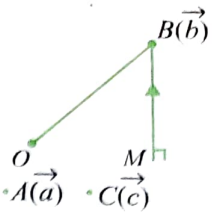
ILLUSTRATION 3.57

Let A, B, C be points with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} + \hat{k}$ and $3\hat{i} + \hat{j} + 2\hat{k}$ respectively. Find the shortest distance between point B and plane OAC.

Sol. Shortest distance between B and plane OAC is perpendicular distance of point B from the plane BM, M is foot of perpendicualr from B on plane OAC.

Now BM = projection vector $\overrightarrow{AB}$ on vector perpendicular to the plane.

Now vector perpendicular to the plane is $\overrightarrow{OA} \times \overrightarrow{OC} = \vec{a} \times \vec{c}$. Therefore,



$$BM = \frac{\overrightarrow{OB} \cdot (\vec{a} \times \vec{c})}{|\vec{a} \times \vec{c}|}$$

$$= \frac{\vec{b} \cdot (\vec{a} \times \vec{c})}{|\vec{a} \times \vec{c}|}$$

Now $$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -3\hat{i} - \hat{j} + 5\hat{k}$$

$\therefore$ $$\vec{b} \cdot (\vec{a} \times \vec{c}) = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (-3\hat{i} - \hat{j} + 5\hat{k}) = -3 - 2 + 15 = 10$$

and $$|\vec{a} \times \vec{c}| = |-3\hat{i} - \hat{j} + 5\hat{k}| = \sqrt{9 + 1 + 25} = \sqrt{35}$$

$\therefore$ $$BM = \frac{10}{\sqrt{35}} = 2\sqrt{\frac{5}{7}}$$

Concept Application Exercise 3.2

1. If $\vec{a} = 2\hat{i} + 3\hat{j} - 5\hat{k}$, $\vec{b} = m\hat{i} + n\hat{j} + 12\hat{k}$ and $\vec{a} \times \vec{b} = \vec{0}$, then find (m, n).

2. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, then find the value of $\vec{a} \cdot \vec{b}$.

3. If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq 0$, where $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors, then for some scalar k prove that $\vec{a} + \vec{c} = k\vec{b}$.

4. If $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} - 4\vec{k}$ and $\vec{c} = \vec{i} + \vec{j} + \vec{k}$, then find the value of $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$.

5. If the vectors $\vec{c}$, $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{b} = \hat{j}$ are such that $\vec{a}, \vec{c}$ and $\vec{b}$ form a right-handed system, then find $\vec{c}$.

6. Given that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a}$ is not a zero vector. Show that $\vec{b} = \vec{c}$.

7. Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$ and give a geometrical interpretation of it.

8. If $\vec{x}$ and $\vec{y}$ are unit vectors and $|\vec{z}| = \frac{2}{\sqrt{7}}$ such that $\vec{z} + \vec{z} \times \vec{x} = \vec{y}$, then find the angle θ between $\vec{x}$ and $\vec{z}$.

9. Prove that $(\vec{a} \cdot \hat{i})(\vec{a} \times \hat{i}) + (\vec{a} \cdot \hat{j})(\vec{a} \times \hat{j}) + (\vec{a} \cdot \hat{k})(\vec{a} \times \hat{k}) = \vec{0}$.

10. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $\lambda\vec{b}\times\vec{a}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$. Then find the value of λ.

11. A particle has an angular speed of 3 rad/s and the axis of rotation passes through the points (1, 1, 2) and (1, 2, – 2). Find the velocity of the particle at point $P(3, 6, 4)$.

12. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}\cdot\vec{b}=0=\vec{a}\cdot\vec{c}$. If the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, then find $\vec{a}$.

13. If $(\vec{a}\times\vec{b})^2+(\vec{a}\cdot\vec{b})^2=144$ and $|\vec{a}|=4$, then find the value of $|\vec{b}|$.

14. Given $|\vec{a}|=|\vec{b}|=1$ and $|\vec{a}+\vec{b}|=\sqrt{3}$. If $\vec{c}$ is a vector such that $\vec{c}-\vec{a}-2\vec{b}=3(\vec{a}\times\vec{b})$, then find the value of $\vec{c}\cdot\vec{b}$.

15. Find the moment of $\vec{F}$ about point (2, –1, 3), when force $\vec{F}=3\hat{i}+2\hat{j}-4\hat{k}$ is acting on point (1, –1, 2).

ANSWERS

1. $m=\frac{-24}{5}, n=\frac{-36}{5}$ **2.** 6 **4.** –74

5. $z\hat{i}-x\hat{k}$ **8.** $\theta=\pi/3=60°$ **10.** $\lambda=2$

11. $\frac{3}{\sqrt{17}}(22\hat{i}-8\hat{j}-2\hat{k})$ **12.** $\pm2(\vec{b}\times\vec{c})$ **13.** $|\vec{b}|=3$

14. $\vec{c}\cdot\vec{b}=5/2$ **15.** $2\hat{i}-7\hat{j}-2\hat{k}$

SCALAR TRIPLE PRODUCT

The **scalar triple product** (also called the **mixed** or **box product**) is defined as the *dot product* of one of the vectors with the *cross product* of the other two.

Thus, scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$.

We denote it by $[\vec{a}\ \vec{b}\ \vec{c}]$.

The scalar triple product can be evaluated numerically using any one of the following equivalent characterizations:

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$$

(The parentheses may be omitted without causing ambiguity, since the *dot product* cannot be evaluated first. If it were, it would leave the cross product of a scalar and a vector, which is not defined.)

10. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $\lambda \vec{b} \times \vec{a} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$. Then find the value of λ.

11. A particle has an angular speed of 3 rad/s and the axis of rotation passes through the points (1, 1, 2) and (1, 2, – 2). Find the velocity of the particle at point $P(3, 6, 4)$.

12. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a} \cdot \vec{b} = 0 = \vec{a} \cdot \vec{c}$. If the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, then find $\vec{a}$.

13. If $(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = 144$ and $|\vec{a}| = 4$, then find the value of $|\vec{b}|$.

14. Given $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = \sqrt{3}$. If $\vec{c}$ is a vector such that $\vec{c} - \vec{a} - 2\vec{b} = 3(\vec{a} \times \vec{b})$, then find the value of $\vec{c} \cdot \vec{b}$.

15. Find the moment of $\vec{F}$ about point (2, –1, 3), when force $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ is acting on point (1, –1, 2).

ANSWERS

1. $m = \frac{-24}{5}, n = \frac{-36}{5}$ **2.** 6 **4.** –74

5. $z\hat{i} - x\hat{k}$ **8.** $\theta = \pi/3 = 60°$ **10.** $\lambda = 2$

11. $\frac{3}{\sqrt{17}}(22\hat{i} - 8\hat{j} - 2\hat{k})$ **12.** $\pm 2(\vec{b} \times \vec{c})$ **13.** $|\vec{b}| = 3$

14. $\vec{c} \cdot \vec{b} = 5/2$ **15.** $2\hat{i} - 7\hat{j} - 2\hat{k}$

SCALAR TRIPLE PRODUCT

The **scalar triple product** (also called the **mixed** or **box product**) is defined as the *dot product* of one of the vectors with the *cross product* of the other two.

Thus, scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$.

We denote it by $[\vec{a}\ \vec{b}\ \vec{c}]$.

The scalar triple product can be evaluated numerically using any one of the following equivalent characterizations:

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \vec{b} \cdot \left(\vec{c} \times \vec{a}\right) = \vec{c} \cdot \left(\vec{a} \times \vec{b}\right)$$

(The parentheses may be omitted without causing ambiguity, since the *dot product* cannot be evaluated first. If it were, it would leave the cross product of a scalar and a vector, which is not defined.)

i.e., $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]=-[\vec{b}\ \vec{a}\ \vec{c}]=-[\vec{c}\ \vec{b}\ \vec{a}]$

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}_2$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}).$$

$$=\begin{vmatrix}\hat{i}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})&\hat{j}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})&\hat{k}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$$

$$=\begin{vmatrix}c_1&c_2&c_3\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$$

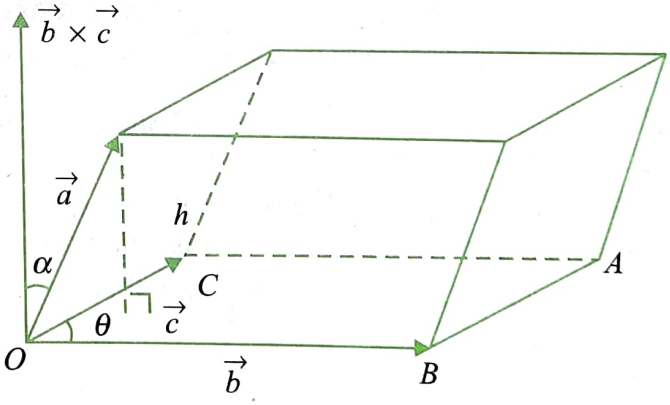
$$=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$$

Also $\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$

$$=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$$

GEOMETRICAL INTERPRETATION

Here $(\vec{a}\times\vec{b})\cdot\vec{c}$ represents (and is equal to) the volume of the parallelepiped whose adjacent sides are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.



$$\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot(bc\sin\theta\,\hat{n})$$

$$=bc\sin\theta\,(\vec{a}\cdot\hat{n})$$

$$=bc\sin\theta\cdot a\cdot 1\cdot\cos\alpha$$

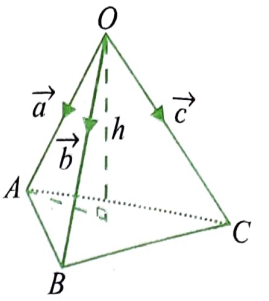
$$=(a\cos\alpha)\,(bc\sin\theta)$$

$$=\text{Height}\times(\text{Area of base})$$

$$=\text{Volume of parallelepiped}$$

VOLUME OF TETRAHEDRON

Tetrahedron is a pyramid having a triangular base. Therefore,



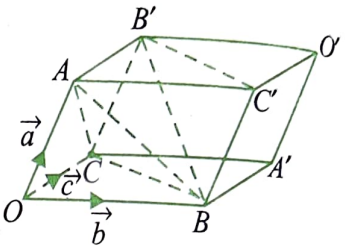
$$\therefore \quad \text{Volume} = \frac{1}{3}(\text{Height}) \times (\text{Area of base})$$

$$= \frac{1}{3}h \cdot \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{6}h\,|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

Now h is projection of $\overrightarrow{OA}$ on vector which is normal to the plane.

Vector normal to the plane ABC is $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$. Therefore,



$$h = \frac{\vec{a} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$

$$= \frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$

$$\therefore \quad \text{Volume} = \frac{1}{6}\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

$$= \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$$

= One sixth the volume of parallelepiped having coterminous vectors $\vec{a}, \vec{b}$ and $\vec{c}$

The figure with four vertices O, A, B and C is a tetrahedron, and its volume is $\frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$ or one sixth of the volume of the parallelepiped. We justify this statement by noting that the parallelepiped can be exactly divided up into six non-overlapping tetrahedral; three of these, with vertices (O, A, B, C), (A, C, B, B') and (A, B, C', B') are shown in figure. If we consider tetrahedron (A, C, B, B'), for example, the vectors along three edges are $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{AC} = \vec{c} - \vec{a}$ and $\overrightarrow{AB'} = \vec{c}$.

Our assumption then gives the volume of this tetrahedron to be $\frac{1}{6}[(\vec{b} - \vec{a})\,(\vec{c} - \vec{a})\,\vec{c}] = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$.

In similar way, the assumed formula gives the same result for the volumes of all six tetrahedral. Since the tetrahedral are non-overlapping, the volume of the parallelepiped should be the sum of the volumes of the tetrahedra; the factor 1/6 ensures that this is so.

PROPERTIES OF SCALAR TRIPLE PRODUCT

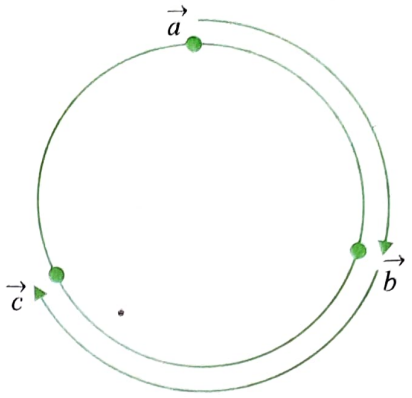
I. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$, i.e., position of the dot and the cross can be interchanged without altering the product.

II. $[k\vec{a}\,\vec{b}\,\vec{c}] = k[\vec{a}\,\vec{b}\,\vec{c}]$ (where k is scalar)

III. $[\vec{a} + \vec{b}\;\vec{c}\;\vec{d}] = [\vec{a}\;\vec{c}\;\vec{d}] + [\vec{b}\;\vec{c}\;\vec{d}]$

IV. $\vec{a}, \vec{b}$ and $\vec{c}$ in that order form a right-handed system if $[\vec{a}\ \vec{b}\ \vec{c}] > 0$;

$\vec{a}, \vec{b}$ and $\vec{c}$ in that order form a left-handed system if $[\vec{a}\ \vec{b}\ \vec{c}] < 0$.



V. The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be coplanar is that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

VI. $[\vec{a}\ \vec{a}\ \vec{b}] = 0 \quad (\because \vec{a}$ is $\perp$ to $\vec{a} \times \vec{b}, \quad \vec{a} \cdot (\vec{a} \times \vec{b}) = 0)$

VII. $[x_1\vec{a} + y_1\vec{b} + z_1\vec{c} \quad x_2\vec{a} + y_2\vec{b} + z_2\vec{c} \quad x_3\vec{a} + y_3\vec{b} + z_3\vec{c}]$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$$

Proof:

$[x_1\vec{a} + y_1\vec{b} + z_1\vec{c} \quad x_2\vec{a} + y_2\vec{b} + z_2\vec{c} \quad x_3\vec{a} + y_3\vec{b} + z_3\vec{c}]$

$= (x_1\vec{a} + y_1\vec{b} + z_1\vec{c}) \cdot (x_2\vec{a} + y_2\vec{b} + z_2\vec{c}) \times (x_3\vec{a} + y_3\vec{b} + z_3\vec{c})$

$= (x_1\vec{a} + y_1\vec{b} + z_1\vec{c}) \cdot [(x_2y_3 - x_3y_2)\vec{a} \times \vec{b} + (y_2z_3 - y_3z_2)\vec{b} \times \vec{c} + (x_3z_2 - x_2z_3)\vec{c} \times \vec{a}]$

$= x_1(y_2z_3 - y_3z_2)\vec{a} \cdot \vec{b} \times \vec{c} + y_1(x_3z_2 - x_2z_3)\ \vec{b} \cdot \vec{c} \times \vec{a} + z_1(x_2y_3 - x_3y_2)\vec{c} \cdot \vec{a} \times \vec{b}$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$$

ILLUSTRATION 3.58

Let $\vec{a} = x\hat{i} + 12\hat{j} - \hat{k}, \vec{b} = 2\hat{i} + 2x\hat{j} + \hat{k}$ and $c = \hat{i} + \hat{k}$. If the ordered set $[\vec{b}\ \vec{c}\ \vec{a}]$ is left handed, then find the values of x.

Sol. For a left-handed system,

$$[\vec{b}\ \vec{c}\ \vec{a}] < 0$$

or
$$\begin{vmatrix} 2 & 2x & 1 \\ 1 & 0 & 1 \\ x & 12 & -1 \end{vmatrix} < 0$$

or
$$2(0 - 12) - 2x(-1 - x) + 1(12) < 0$$

or
$$x^2 + x - 6 < 0$$

or
$$x \in (-3, 2)$$

ILLUSTRATION 3.59

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then find the value of

$$\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\vec{b}\cdot(\vec{c}\times\vec{a})}+\frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\vec{c}\cdot(\vec{a}\times\vec{b})}+\frac{\vec{c}\cdot(\vec{b}\times\vec{a})}{\vec{a}\cdot(\vec{b}\times\vec{c})}.$$

Sol. Since $[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$, we have

$$\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\vec{b}\cdot(\vec{c}\times\vec{a})}+\frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\vec{c}\cdot(\vec{a}\times\vec{b})}+\frac{\vec{c}\cdot(\vec{b}\times\vec{a})}{\vec{a}\cdot(\vec{b}\times\vec{c})}=\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{b}\ \vec{c}\ \vec{a}]}+\frac{[\vec{b}\ \vec{c}\ \vec{a}]}{[\vec{c}\ \vec{a}\ \vec{b}]}+\frac{[\vec{c}\ \vec{b}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$=\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]}+\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]}-\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$=1+1-1=1$$

ILLUSTRATION 3.60

If the vectors $2\hat{i}-3\hat{j}$, $\hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}-\hat{k}$ form three concurrent edges of a parallelepiped, then find the volume of the parallelepiped.

Sol. Here, $\overrightarrow{OA}=2\hat{i}-3\hat{j}=\vec{a}$ (say),

$\overrightarrow{OB}=\hat{i}+\hat{j}-\hat{k}=\vec{b}$ (say),

and $\overrightarrow{OC}=3\hat{i}-\hat{k}=\vec{c}$ (say)

Hence, Volume $=[\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix}2&-3&0\\1&1&-1\\3&0&-1\end{vmatrix}=4$

ILLUSTRATION 3.61

The position vectors of the four angular points of a tetrahedron are $A(\hat{j}+2\hat{k})$, $B(3\hat{i}+\hat{k})$, $C(4\hat{i}+3\hat{j}+6\hat{k})$ and $D(2\hat{i}+3\hat{j}+2\hat{k})$. Find the volume of the tetrahedron $ABCD$.

Sol. Vertices of tetrahedron are $A(\hat{j}+2\hat{k})$, $B(3\hat{i}+\hat{k})$, $C(4\hat{i}+3\hat{j}+6\hat{k})$ and $D(2\hat{i}+3\hat{j}+2\hat{k})$. Therefore,

$\overrightarrow{AB}=3\hat{i}-\hat{j}-\hat{k}$, $\overrightarrow{AC}=4\hat{i}+2\hat{j}+4\hat{k}$ and $\overrightarrow{AD}=2\hat{i}+2\hat{j}$

$\therefore$ Volume, $V=\frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$$=\frac{1}{6}\begin{vmatrix}3&-1&-1\\4&2&4\\2&2&0\end{vmatrix}$$

$=2$ cubic unit

ILLUSTRATION 3.62

Let $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors and $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$. If the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$, then find the value of $|[\vec{a}\ \vec{b}\ \vec{c}]|$.

Sol. $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$

$\vec{a}$ is perpendicular to vectors $\vec{b}$ and $\vec{c}$. Thus,

$$\vec{a} = \lambda(\vec{b} \times \vec{c})$$

or
$$|\vec{a}| = |\lambda(\vec{b} \times \vec{c})| = \left|\lambda \frac{\sqrt{3}}{2}\right| = 1$$

Now
$$|[\vec{a}\ \vec{b}\ \vec{c}]| = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$
$$= |\lambda||(\vec{b} \times \vec{c})|^2$$
$$= \frac{2}{\sqrt{3}}|\vec{b}|^2 |\vec{c}|^2 \sin^2\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}} \times \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{\sqrt{3}}{2}$$

ILLUSTRATION 3.63

Prove that $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2\,[\vec{a}\ \ \vec{b}\ \ \vec{c}]$.

Sol. $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = (\vec{a}+\vec{b}) \cdot ((\vec{b}+\vec{c}) \times (\vec{c}+\vec{a}))$

$$= (\vec{a}+\vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a})$$
$$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{b}\ \vec{c}\ \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$

ILLUSTRATION 3.64

Prove that $[\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c} \end{vmatrix}$.

Sol. Let $\vec{l} = l_1\hat{i} + l_2\hat{j} + l_3\hat{k}$, $\vec{m} = m_1\hat{i} + m_2\hat{j} + m_3\hat{k}$ and $\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$\therefore$
$$\vec{l} \cdot \vec{a} = l_1a_1 + l_2a_2 + l_3a_3 = \Sigma l_1a_1$$

Similarly,
$$\vec{l} \cdot \vec{b} = \Sigma l_1b_1, \text{ etc.}$$

Now $$[\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma l_1 a_1 & \Sigma l_1 b_1 & \Sigma l_1 c_1 \\ \Sigma m_1 a_1 & \Sigma m_1 b_1 & \Sigma m_1 c_1 \\ \Sigma n_1 a_1 & \Sigma n_1 b_1 & \Sigma n_1 c_1 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$$

ILLUSTRATION 3.65

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}, \vec{c} = \hat{i} + 2\hat{j} - \hat{k}$, then find the value of $\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$.

Sol. $$\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}.\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

Now $$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 4$$

$\therefore$ $$\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = 4^2 = 16$$

ILLUSTRATION 3.66

Find the value of a so that the volume of the parallelepiped formed by vectors $\hat{i} + a\hat{j} + \hat{k}$, $\hat{j} + a\hat{k}$ and $a\hat{i} + \hat{k}$ becomes minimum.

Sol. $$V = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 1 - a + a^3$$

$\Rightarrow$ $$\frac{dV}{da} = 3a^2 - 1$$

V is minimum at $a = \dfrac{1}{\sqrt{3}}$

ILLUSTRATION 3.67

If $\vec{u}, \vec{v}$ and $\vec{w}$ are three non-coplanar vectors, then prove that

$$(\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}-\vec{v})\times(\vec{v}-\vec{w}) = \vec{u}\cdot\vec{v}\times\vec{w}$$

Sol.
$$(\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}-\vec{v})\times(\vec{v}-\vec{w}) = (\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}\times\vec{v}-\vec{u}\times\vec{w}-\vec{v}\times\vec{v}+\vec{v}\times\vec{w})$$
$$= (\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}\times\vec{v}-\vec{u}\times\vec{w}+\vec{v}\times\vec{w})$$
$$= 0-0+\vec{u}\cdot(\vec{v}\times\vec{w})+0-\vec{v}\cdot(\vec{u}\times\vec{w})+0-\vec{w}\cdot(\vec{u}\times\vec{v})+0-0$$
$$= [\vec{u}\ \vec{v}\ \vec{w}] + [\vec{v}\ \vec{w}\ \vec{u}] - [\vec{w}\ \vec{u}\ \vec{v}] = \vec{u}\cdot(\vec{v}\times\vec{w})$$

ILLUSTRATION 3.68

If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\times\vec{b}| = 2$, then find the value of $[\vec{a}\ \vec{b}\ \vec{a}\times\vec{b}]$.

Sol.
$$[\vec{a}\ \vec{b}\ \vec{a}\times\vec{b}] = (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$$
$$= |\vec{a}\times\vec{b}|^2$$
$$= 4$$

ILLUSTRATION 3.69

Find the altitude of a parallelepiped whose three coterminous edges are vectors $\vec{A} = \hat{i}+\hat{j}+\hat{k}$, $\vec{B} = 2\hat{i}+4\hat{j}-\hat{k}$ and $\vec{C} = \hat{i}+\hat{j}+3\hat{k}$ with $\vec{A}$ and $\vec{B}$ as the sides of the base of the parallelepiped.

Sol.
$$h = \frac{\text{Volume of parallelepiped}}{\text{Area of base}}$$
$$= \frac{[\vec{A}\ \vec{B}\ \vec{C}]}{|\vec{A}\times\vec{B}|} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & -1 \end{vmatrix}} = \frac{4}{|-5\hat{i}+3\hat{j}+2\hat{k}|} = \frac{2\sqrt{38}}{19}$$

ILLUSTRATION 3.70

If $[\vec{a}\,\vec{b}\,\vec{c}] = 2$, then find the value of $[(\vec{a}+2\vec{b}-\vec{c})(\vec{a}-\vec{b})(\vec{a}-\vec{b}-\vec{c})]$.

Sol. $[(\vec{a}+2\vec{b}-\vec{c})(\vec{a}-\vec{b})(\vec{a}-\vec{b}-\vec{c})]$

$$= \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{vmatrix} [\vec{a}\,\vec{b}\,\vec{c}]$$

$$= 3[\vec{a}\,\vec{b}\,\vec{c}] = 6$$

ILLUSTRATION 3.71

If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors and $\vec{a} = \alpha(\vec{a} \times \vec{b}) + \beta(\vec{b} \times \vec{c}) + \gamma(\vec{c} \times \vec{a})$ and $[\vec{a}\,\vec{b}\,\vec{c}] = 1$, then find the value of $\alpha + \beta + \gamma$.

Sol. Taking dot product with $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, we get

$$|\vec{a}|^2 = \beta \cdot [\vec{a}\,\vec{b}\,\vec{c}] = \beta$$

$$0 = \gamma \cdot [\vec{a}\,\vec{b}\,\vec{c}] = \gamma$$

and $$0 = \alpha \cdot [\vec{a}\,\vec{b}\,\vec{c}] = \alpha$$

$\therefore$ $$\alpha + \beta + \gamma = |\vec{a}|^2$$

ILLUSTRATION 3.72

If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar vectors, then prove that $|(\vec{a} \cdot \vec{d})(\vec{b} \times \vec{c}) + (\vec{b} \cdot \vec{d})(\vec{c} \times \vec{a}) + (\vec{c} \cdot \vec{d})(\vec{a} \times \vec{b})|$ is independent of $\vec{d}$, where $\vec{d}$ is a unit vector.

Sol. Given $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$ as $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar. Also there does not exist any linear relation between them because if any such relation exists, then they would be coplanar.

Let $$A = x(\vec{b} \times \vec{c}) + y(\vec{c} \times \vec{a}) + z(\vec{a} \times \vec{b}),$$

where $$x = \vec{a} \cdot \vec{d}, y = \vec{b} \cdot \vec{d}, z = \vec{c} \cdot \vec{d}$$

We have to find the value of modulus of $\vec{A}$, i.e., $|\vec{A}|$, which is independent of $\vec{d}$.

Multiplying both sides scalarly by $\vec{a}, \vec{b}$ and $\vec{c}$, and knowing that scalar triple product is zero when two vectors are equal, we get

$$\vec{A} \cdot \vec{a} = x[\vec{a}\,\vec{b}\,\vec{c}] + 0$$

Putting for x, we get

$$(\vec{a} \cdot \vec{d})[\vec{a}\,\vec{b}\,\vec{c}] = \vec{A} \cdot \vec{a}$$

Similarly, we have

$$(\vec{b} \cdot \vec{d})[\vec{a}\,\vec{b}\,\vec{c}] = \vec{A} \cdot \vec{b}$$

$$(\vec{c} \cdot \vec{d})[\vec{a}\,\vec{b}\,\vec{c}] = \vec{A} \cdot \vec{c}$$

Adding the above relations, we get

$$[(\vec{a}+\vec{b}+\vec{c})\cdot\vec{d}]\ [\vec{a}\ \vec{b}\ \vec{c}] = \vec{A}\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$(\vec{a}+\vec{b}+\vec{c})\cdot[\vec{d}\ [\vec{a}\ \vec{b}\ \vec{c}] - A\vec{A} = 0$$

or

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, $\vec{a}+\vec{b}+\vec{c} \neq 0$ because otherwise any one is expressible as a linear combination of other two. Hence,

$$[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = \vec{A}$$

$$|\vec{A}| = |[\vec{a}\ \vec{b}\ \vec{c}]| \text{ as } \vec{d} \text{ is a unit vector.}$$

It is independent of $\vec{d}$.

ILLUSTRATION 3.73

Prove that vectors

$$\vec{u} = (al + a_1l_1)\,\hat{i} + (am + a_1m_1)\,\hat{j} + (an + a_1n_1)\,\hat{k}$$

$$\vec{v} = (bl + b_1l_1)\,\hat{i} + (bm + b_1m_1)\,\hat{j} + (bn + b_1n_1)\,\hat{k}$$

$$\vec{w} = (cl + c_1l_1)\,\hat{i} + (cm + c_1m_1)\,\hat{j} + (cn + c_1n_1)\,\hat{k}$$

are coplanar.

Sol.

$$[\vec{u}\ \vec{v}\ \vec{w}] = \begin{vmatrix} al+a_1l_1 & am+a_1m_1 & an+a_1n_1 \\ bl+b_1l_1 & bm+b_1m_1 & bn+b_1n_1 \\ cl+c_1l_1 & cm+c_1m_1 & cn+c_1n_1 \end{vmatrix}$$

$$= \begin{vmatrix} a & a_1 & 0 \\ b & b_1 & 0 \\ c & c_1 & 0 \end{vmatrix} \begin{vmatrix} l & l_1 & 0 \\ m & m_1 & 0 \\ n & n_1 & 0 \end{vmatrix} = 0$$

Therefore, the given vectors are coplanar.

ILLUSTRATION 3.74

Let G_1, G_2 and G_3 be the centroids of the trianglular faces OBC, OCA and OAB, respectively, of a tetrahedron $OABC$. If V_1 denotes the volume of the tetrahedron $OABC$ and V_2 that of the parallelepiped with OG_1, OG_2 and OG_3 as three concurrent edges, then prove that $4V_1 = 9V_1$.

Sol. Taking O as the origin, let the position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. Then the position vectors G_1, G_2 and G_3 are $\dfrac{\vec{b}+\vec{c}}{3}$, $\dfrac{\vec{c}+\vec{a}}{3}$ and $\dfrac{\vec{a}+\vec{b}}{3}$, respectively. Therefore,

$$V_1 = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}] \text{ and } V_2 = [\overrightarrow{OG_1}\ \overrightarrow{OG_2}\ \overrightarrow{OG_3}]$$

Now, $$V_2 = [\overrightarrow{OG_1}\ \overrightarrow{OG_2}\ \overrightarrow{OG_3}]$$

$$= \frac{1}{27}[\vec{b}+\vec{c}\ \vec{c}+\vec{a}\ \vec{a}+\vec{b}]$$

$$= \frac{2}{27}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$= \frac{2}{27} \times 6V_1 \text{ or } 9V_2 = 4V_1$$

VECTOR TRIPLE PRODUCT

The vector triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is the vector

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Also $$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

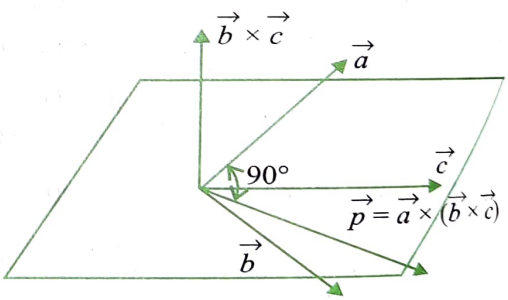
In general, $$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, then the vectors $\vec{a}$ and $\vec{c}$ are collinear.

$\vec{p} = \vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to $\vec{a}$ and $\vec{b} \times \vec{c}$, but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

Hence, vector $\vec{p}$ must lie in the plane of $\vec{b}$ and $\vec{c}$.

$$\Rightarrow \quad \vec{p} = \vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c} \qquad \text{(i)}$$



Multiplying (i) scalarly by $\vec{a}$, we have

$$\vec{p} \cdot \vec{a} = x(\vec{a} \cdot \vec{b}) + y(\vec{a} \cdot \vec{c}) \qquad \text{(ii)}$$

But $\vec{p} \perp \vec{a} \Rightarrow \vec{p} \cdot \vec{a} = 0$. Therefore,

$$x(\vec{a} \cdot \vec{b}) = -y(\vec{c} \cdot \vec{a}), \text{ i.e., } \frac{x}{\vec{c} \cdot \vec{a}} = \frac{-y}{\vec{a} \cdot \vec{b}} = \lambda$$

$$\therefore \quad x = \lambda(\vec{c} \cdot \vec{a}),\ y = -\lambda(\vec{a} \cdot \vec{b}) \qquad \text{(iii)}$$

Substituting x and y from (iii) in (i), $\vec{a} \times (\vec{b} \times \vec{c}) = \lambda[(\vec{c} \cdot \vec{a})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}]$ (iv)

The simplest way to determine λ is by taking specific vectors $\vec{a} = \hat{i}, \vec{b} = \hat{i}, \vec{c} = \hat{j}$

We have from (iv), $\hat{i} \times (\hat{i} \times \hat{j}) = \lambda[(\hat{i} \cdot \hat{j})\hat{i} - (\hat{i} \cdot \hat{i})\hat{j}]$, i.e., $\hat{i} \times \hat{k} = \lambda[0\hat{i} - 1\hat{j}]$, i.e., $-\hat{j} = -\lambda\hat{j}$

$\lambda = 1$

$\therefore$ Substituting λ in (iv), $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

LAGRANGE'S IDENTITY

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})]$$
$$= \vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}]$$
$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$
$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

This is called Lagrange's identity.

Note:

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [(\vec{a}\times\vec{b})\cdot\vec{d}]\vec{c} - [(\vec{a}\times\vec{b})\cdot\vec{c}]\vec{d} = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Thus, vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ lies in the plane of $\vec{c}$ and $\vec{d}$; otherwise

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = -[(\vec{c}\times\vec{d})\cdot\vec{b}]\vec{a} + [(\vec{c}\times\vec{d})\vec{a}]\vec{b}$$

which shows that the vector lies in the plane of $\vec{a}$ and $\vec{b}$. Thus, the vector lies along the common section of the plane of $\vec{c}$ and $\vec{d}$ and the plane of $\vec{a}$ and $\vec{b}$.

ILLUSTRATION 3.75

Prove that $\hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$.

Sol. $\hat{i}\times(\vec{a}\times\hat{i}) = (\hat{i}\cdot\hat{i})\vec{a} - (\vec{a}\cdot\hat{i})\hat{i} = \vec{a} - (\vec{a}\cdot\hat{i})\hat{i}$

Similarly, $\hat{j}\times(\vec{a}\times\hat{j}) = \vec{a} - (\vec{a}\cdot\hat{j})\hat{j}$ and $\hat{k}\times(\vec{a}\times\hat{k}) = \vec{a} - (\vec{a}\cdot\hat{k})\hat{k}$.

$\therefore$ $\hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k}) = 3\vec{a} - ((\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\hat{j} + (\vec{a}\times\hat{k})\hat{k}) = 2\vec{a}$

ILLUSTRATION 3.76

If $\hat{i}\times[(\vec{a}-\hat{j})\times\hat{i}] + \hat{j}\times[(\vec{a}-\hat{k})\times\hat{j}] + \hat{k}\times[(\vec{a}-\hat{i})\times\hat{k}] = 0$, then find vector $\vec{a}$.

Sol. $\hat{i}\times[(\vec{a}-\hat{j})\times\hat{i}] = (\hat{i}\cdot\hat{i})(\vec{a}-\hat{j}) - (\hat{i}\cdot(\vec{a}-\hat{j}))\hat{i}$

$$= \vec{a} - \hat{j} - (\hat{i}\cdot\vec{a})\hat{i}$$

Similarly, $\hat{j}\times[(\vec{a}-\hat{k})\times\hat{j}] = \vec{a} - \hat{k} + (\hat{j}\cdot\vec{a})\hat{j}$

And $\hat{k}\times[(\vec{a}-\hat{i})\times\hat{k}]=\vec{a}-\hat{i}-(\hat{k}\cdot\vec{a})\hat{k}$

$\therefore$ $\hat{i}\times[(\vec{a}-\hat{j})\times\hat{i}]+\hat{j}\times[(\vec{a}-\hat{k})\times\hat{j}]+\hat{k}\times[(\vec{a}-\hat{i})\times\hat{k}]$

$=\vec{a}-\hat{j}-(\hat{i}\cdot\vec{a})\hat{i}+\vec{a}-\hat{k}+(\hat{j}\cdot\vec{a})\hat{j}+\vec{a}-\hat{i}-(\hat{k}\cdot\vec{a})\hat{k}=0$

$\therefore$ $3\vec{a}-(\hat{i}+\hat{j}+\hat{k})-\vec{a}=0$

or $\vec{a}=\frac{1}{2}(\hat{i}+\hat{j}+\hat{k})$

ILLUSTRATION 3.77

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then prove that $[\vec{a}\times\vec{b}\quad\vec{b}\times\vec{c}\quad\vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]^2$.

Sol. $[\vec{a}\times\vec{b}\quad\vec{b}\times\vec{c}\quad\vec{c}\times\vec{a}]=(\vec{a}\times\vec{b})\cdot((\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}))$

$=(\vec{a}\times\vec{b})\cdot[[\vec{b}\ \vec{c}\ \vec{a}]\vec{c}-[\vec{b}\ \vec{c}\ \vec{c}]\vec{a}]$

$=[\vec{a}\ \vec{b}\ \vec{c}]^2$

ILLUSTRATION 3.78

For any four vectors, prove that

$$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0.$$

Sol. $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})=(\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d})-(\vec{b}\cdot\vec{d})(\vec{c}\cdot\vec{a})$

$(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})=(\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d})-(\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b})$

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$\Rightarrow$ $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0$

ILLUSTRATION 3.79

If $\vec{b}$ and $\vec{c}$ are two non-collinear vectors such that $\vec{a}\parallel(\vec{b}\times\vec{c})$, then prove that $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})$ is equal to $|\vec{a}|^2(\vec{b}\cdot\vec{c})$.

Sol. $\vec{a}\parallel(\vec{b}\times\vec{c})$

$\Rightarrow$ $\vec{a}=\lambda(\vec{b}\times\vec{c})$ and $\vec{a}=\vec{a}\perp\vec{b}$ and $\vec{a}\perp\vec{c}$

Now $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{c}\end{vmatrix}$

$=\begin{vmatrix}\vec{a}.\vec{a} & 0\\ 0 & \vec{b}.\vec{c}\end{vmatrix}=|\vec{a}|^2(\vec{b}\cdot\vec{c})$

ILLUSTRATION 3.80

Find the vector of length 3 unit which is perpendicular to $\hat{i}+\hat{j}+\hat{k}$ and lies in the plane of $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-3\hat{j}$.

Sol. Given vectors are

$$\vec{a}=\hat{i}+\hat{j}+\hat{k}$$

$$\vec{b}=\hat{i}+\hat{j}+\hat{k}$$

and

$$\vec{c}=2\hat{i}-3\hat{j}$$

Vector perpendicular to vectors $\vec{b}$ and $\vec{c}$ is $\vec{b}\times\vec{c}$.

Vector perpendicular to $\vec{a}$ and $\vec{b}\times\vec{c}$ is $\vec{a}\times(\vec{b}\times\vec{c})$ which lies in the plane of $\vec{b}$ and $\vec{c}$. Therefore,

$$\vec{a}\times(\vec{b}\times\vec{c})=[(\hat{i}+\hat{j}+\hat{k})\times\{(\hat{i}+\hat{j}+\hat{k})\times(2\hat{i}-3\hat{j})\}]$$

$$=-(\hat{i}+\hat{j}+\hat{k})-3(2\hat{i}-3\hat{j})$$

$$=-7\hat{i}+8\hat{j}-\hat{k}$$

$$\therefore\quad \frac{\vec{a}\times(\vec{b}\times\vec{c})}{|\vec{a}\times(\vec{b}\times\vec{c})|}=\frac{-7\hat{i}+8\hat{j}-\hat{k}}{\sqrt{114}}$$

$$\therefore\quad \text{Required vector}=3\frac{\vec{a}\times(\vec{b}\times\vec{c})}{|\vec{a}\times(\vec{b}\times\vec{c})|}=3\frac{-7\hat{i}+8\hat{j}-\hat{k}}{\sqrt{114}}$$

ILLUSTRATION 3.81

Let $\hat{a},\hat{b}$ and $\hat{c}$ be the non-coplanar unit vectors. The angle between $\hat{b}$ and $\hat{c}$ is α, between $\hat{c}$ and $\hat{a}$ is β and between $\hat{a}$ and $\hat{b}$ is γ. If $A(\hat{a}\cos\alpha), B(\hat{b}\cos\beta)$ and $C(\hat{c}\cos\gamma)$, then show that in triangle ABC, $\frac{|\hat{a}\times(\hat{b}\times\hat{c})|}{\sin A}=\frac{|\hat{b}\times(\hat{c}\times\hat{a})|}{\sin B}=\frac{|\hat{c}\times(\hat{a}\times\hat{b})|}{\sin C}=\frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{|\sum\sin\alpha\cdot\cos\beta\cdot\cos\gamma\hat{n}_1|}$, where $\hat{n}_1=\frac{\hat{b}\times\hat{c}}{|\hat{b}\times\hat{c}|}$, $\hat{n}_2=\frac{\hat{c}\times\hat{a}}{|\hat{c}\times\hat{a}|}$ and $\hat{n}_3=\frac{\hat{a}\times\hat{b}}{|\hat{a}\times\hat{b}|}$.

Sol. From the sine rule, we get

$$\frac{AB}{\sin C}=\frac{AC}{\sin B}=\frac{BC}{\sin A}=\frac{(AB)(BC)(CA)}{2\Delta ABC}$$

$$BC=|\overrightarrow{BC}|=|\hat{c}\cos\gamma-\hat{b}\cos\beta|=|(\hat{a}\cdot\hat{b})\hat{c}-(\hat{c}\cdot\hat{a})\hat{b}|=|(\hat{a}\times(\hat{b}\times\hat{c}))|$$

Similarly,

$$AC=|\overrightarrow{AC}|=|\hat{b}\times(\hat{c}\times\hat{a})|\text{ and }AB=|\overrightarrow{AB}|=|\hat{c}\times(\hat{a}\times\hat{b})|$$

Also,

$$\Delta ABC = \frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$$
$$= \frac{1}{2}|(\hat{c}\cos\gamma - \hat{b}\cos\beta)\times(\hat{a}\cos\alpha - \hat{b}\cos\beta)|$$
$$= \frac{1}{2}|(\hat{c}\times\hat{a})\cos\alpha\cos\gamma + (\hat{b}\times\hat{c})\cos\gamma\cos\beta + (\hat{a}\times\hat{b})\cos\beta\cos\alpha|$$

or $$2\Delta ABC = \left|\sum \hat{n}_1 \sin\alpha\cos\beta\cos\gamma\right|$$

$\Rightarrow$ $$\frac{|\hat{a}\times(\hat{b}\times\hat{c})|}{\sin A} = \frac{|\hat{b}\times(\hat{c}\times\hat{a})|}{\sin B} = \frac{|\hat{c}\times(\hat{a}\times\hat{b})|}{\sin C} = \frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{\left|\sum \sin\alpha\cos\beta\cos\gamma\,\hat{n}_1\right|}$$

ILLUSTRATION 3.82

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then prove that

$$\vec{d} = \frac{\vec{a}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{b}\times\vec{c}) + \frac{\vec{b}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{c}\times\vec{a}) + \frac{\vec{c}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{a}\times\vec{b})$$

Sol. Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, vectors $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are also non-coplanar. Let

$$\vec{d} = l(\vec{b}\times\vec{c}) + m(\vec{c}\times\vec{a}) + n(\vec{a}\times\vec{b}) \quad \text{(i)}$$

Now multiplying both sides of (i) scalarly by $\vec{a}$, we have

$$\vec{a}\cdot\vec{d} = l\,\vec{a}\cdot(\vec{b}\times\vec{c}) + m\,\vec{a}\cdot(\vec{c}\times\vec{a}) + n\,\vec{a}\cdot(\vec{a}\times\vec{b}) = l[\vec{a}\ \vec{b}\ \vec{c}] \quad (\because [\vec{a}\ \vec{c}\ \vec{a}] = 0 = [\vec{a}\ \vec{a}\ \vec{b}])$$

$\Rightarrow$ $$l = (\vec{a}\cdot\vec{d})/[\vec{a}\ \vec{b}\ \vec{c}]$$

Similarly, multiplying (i) scalarly by $\vec{b}$ and $\vec{c}$ successively, we get

$$m = (\vec{b}\cdot\vec{d})/[\vec{a}\ \vec{b}\ \vec{c}] \text{ and } n = (\vec{c}\cdot\vec{d})/[\vec{a}\ \vec{b}\ \vec{c}]$$

Putting these values of l, m and n in (i), we get the required relation.

ILLUSTRATION 3.83

If $\vec{b}$ is not perpendicular to $\vec{c}$, then find the vector $\vec{r}$ satisfying the equation $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$ and $\vec{r}\cdot\vec{c} = 0$.

Sol. Given $\vec{r}\times\vec{b} = \vec{a}\times\vec{b} \Rightarrow (\vec{r}-\vec{a})\times\vec{b} = 0$

Hence, $(\vec{r}-\vec{a})$ and $\vec{b}$ are parallel.

$\Rightarrow$ $$\vec{r}-\vec{a} = t\vec{b} \quad \text{(i)}$$

Also $\vec{r}\cdot\vec{c}=0$

Therefore, Taking dot product of (i) by $\vec{c}$, we get $\vec{r}\cdot\vec{c}-\vec{a}\cdot\vec{c}=t(\vec{b}\cdot\vec{c})$

or $$0-\vec{a}\cdot\vec{c}=t(\vec{b}\cdot\vec{c}) \quad \text{or} \quad t=-\left(\frac{\vec{a}\cdot\vec{c}}{\vec{b}\cdot\vec{c}}\right) \tag{ii}$$

From (i) and (ii), solution of $\vec{r}$ is $\vec{r}=\vec{a}-\left(\frac{\vec{a}\cdot\vec{c}}{\vec{b}\cdot\vec{c}}\right)\vec{b}$

ILLUSTRATION 3.84

If $\vec{a}$ and $\vec{b}$ are two given vectors and k is any scalar, then find the vector $\vec{r}$ satisfying $\vec{r}\times\vec{a}+k\vec{r}=\vec{b}$.

Sol. $$\vec{r}\times\vec{a}+k\vec{r}=\vec{b} \tag{i}$$

or $$(\vec{r}\times\vec{a})\times\vec{a}+k\vec{r}\times\vec{a}=\vec{b}\times\vec{a}$$

or $$(\vec{r}\cdot\vec{a})\vec{a}-(\vec{a}\cdot\vec{a})\vec{r}+k(\vec{b}-k\vec{r})=\vec{b}\times\vec{a}$$

or $$(\vec{r}\cdot\vec{a})\vec{a}+k\vec{b}-\vec{b}\times\vec{a}=(|\vec{a}|^2+k^2)\vec{r}$$

or $$\vec{r}=\frac{(\vec{r}\cdot\vec{a})\vec{a}+k\vec{b}-\vec{b}\times\vec{a}}{|\vec{a}|^2+k^2}$$

Also in Eq. (i), taking dot product with $\vec{a}$, we have

$$(\vec{r}\times\vec{a})\cdot\vec{a}+k\vec{r}\cdot\vec{a}=\vec{b}\cdot\vec{a}$$

or $$\vec{r}\cdot\vec{a}=\frac{\vec{b}\cdot\vec{a}}{k}$$

$\Rightarrow$ $$\vec{r}=\frac{1}{k^2+|\vec{a}|^2}\left[\frac{(\vec{a}\cdot\vec{b})\vec{a}}{k}+k\vec{b}+(\vec{a}\times\vec{b})\right]$$

ILLUSTRATION 3.85

If $\vec{r}\cdot\vec{a}=0$, $\vec{r}\cdot\vec{b}=1$ and $[\vec{r}\ \vec{a}\ \vec{b}]=1$, $\vec{a}\cdot\vec{b}\neq 0$, $(\vec{a}\cdot\vec{b})^2-|\vec{a}|^2|\vec{b}|^2=1$, then find $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$.

Sol. Writing $\vec{r}$ as a linear combination of $\vec{a}$, $\vec{b}$ and $\vec{a}\times\vec{b}$, we have

$$\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$$

For scalars x, y and z,

$$0 = \vec{r}\cdot\vec{a} = x|\vec{a}|^2 + y\,\vec{a}\cdot\vec{b} \quad \text{(taking dot product with } \vec{a}\text{)}$$

$$1 = \vec{r}\cdot\vec{b} = x\,\vec{a}\cdot\vec{b} + y|\vec{b}|^2 \quad \text{(taking dot product with } \vec{b}\text{)}$$

Solving, we get $y = \dfrac{|\vec{a}|^2}{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2} = |\vec{a}|^2$

and $$x = \frac{\vec{a}\cdot\vec{b}}{(\vec{a}\cdot\vec{b})^2 - |\vec{a}|^2|\vec{b}|^2} = \vec{a}\cdot\vec{b}$$

Also $$1 = [\vec{r}\,\vec{a}\,\vec{b}] = z|\vec{a}\times\vec{b}|^2 \quad \text{(taking dot product with } \vec{a}\times\vec{b}\text{)}$$

or $$z = \frac{1}{|a\times b|^2}$$

Thus, $$\vec{r} = ((\vec{a}\cdot\vec{b})\vec{a} - |\vec{a}|^2\vec{b}) + \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$$

$$= \vec{a}\times(\vec{a}\times\vec{b}) + \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$$

ILLUSTRATION 3.86

If vector $\vec{x}$ satisfying $\vec{x}\times\vec{a} + (\vec{x}\cdot\vec{b})\vec{c} = \vec{d}$ is given by $\vec{x} = \lambda\vec{a} + \vec{a}\times\dfrac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}\cdot\vec{c})|\vec{a}|^2}$, then find the value of λ.

Sol. $\vec{x}\times\vec{a} + (\vec{x}\cdot\vec{b})\vec{c} = \vec{d}$

$\therefore$ $$\{\vec{x}\times\vec{a} + (\vec{x}\cdot\vec{b})\vec{c}\}\times\vec{c} = \vec{d}\times\vec{c}$$

or $$(\vec{x}\times\vec{a})\times\vec{c} + (\vec{x}\cdot\vec{b})(\vec{c}\times\vec{c}) = \vec{d}\times\vec{c}$$

or $$(\vec{x}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{x} = (\vec{d}\times\vec{c})$$

or $$\vec{a}\times\{(\vec{x}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{x}\} = \vec{a}\times(\vec{d}\times\vec{c})$$

or $$-(\vec{a}\cdot\vec{c})(\vec{a}\times\vec{x}) = \vec{a}\times(\vec{d}\times\vec{c}) \quad (\because \vec{a}\times\vec{a} = 0)$$

or $$\vec{x}\times\vec{a} = \frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

or $$\vec{a}\times(\vec{x}\times\vec{a}) = \vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$(\vec{a}\cdot\vec{a})\vec{x}-(\vec{a}\cdot\vec{x})\vec{a}=\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

or

$$(\vec{a}\cdot\vec{a})\vec{x}=(\vec{a}\cdot\vec{x})\vec{a}+\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

or

$$\vec{x}=\frac{(\vec{a}\cdot\vec{x})\vec{a}}{|\vec{a}|^2}+\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}\cdot\vec{c})|\vec{a}|^2},\ \text{where}\ \lambda=\frac{\vec{a}\cdot\vec{x}}{|\vec{a}|^2}$$

or

ILLUSTRATION 3.87

$\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors and $\vec{r}$ is any arbitrary vector. Prove that $[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$.

Sol. Let $\vec{r}=x_1\vec{a}+x_2\vec{b}+x_3\vec{c}\Rightarrow\vec{r}\cdot(\vec{b}\times\vec{c})=x_1\vec{a}\cdot(\vec{b}\times\vec{c})$ or $x_1=\dfrac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Also, $\vec{r}\cdot(\vec{c}\times\vec{a})=x_2\vec{b}\cdot(\vec{c}\times\vec{a})$ or $x_2=\dfrac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$

and $\vec{r}\cdot(\vec{a}\times\vec{b})=x_3\vec{c}\cdot(\vec{a}\times\vec{b})$ or $x_3=\dfrac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$$\Rightarrow\quad\vec{r}=\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{a}+\frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{b}+\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{c}$$

or $$[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$$

ILLUSTRATION 3.88

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$, $\vec{b}$ and $\vec{c}$ are non-parallel, then prove that the angle between $\vec{a}$ and $\vec{b}$ is $3\pi/4$.

Sol. $\vec{a}\times(\vec{b}\times\vec{c})=\dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$

$$\Rightarrow\quad(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=\frac{1}{\sqrt{2}}\vec{b}+\frac{1}{\sqrt{2}}\vec{c}\qquad\text{(i)}$$

Since $\vec{b}$ and $\vec{c}$ are non-collinear, comparing coefficients of $\vec{c}$ on both sides of (i), we get

$$-\vec{a}\cdot\vec{b}=\frac{1}{\sqrt{2}}\quad\text{or}\quad\vec{a}\cdot\vec{b}=-\frac{1}{\sqrt{2}}$$

$$\Rightarrow \qquad (1)\,(1)\cos\theta = -\frac{1}{\sqrt{2}}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. Therefore,

$$\cos\theta = -\frac{1}{\sqrt{2}} \quad \text{or} \quad \cos\theta = \cos 135^\circ \quad \text{or} \quad \theta = 135^\circ = 3\pi/4$$

ILLUSTRATION 3.89

Prove that $\vec{R} + \dfrac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\,\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2} + \dfrac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\,\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2} = \dfrac{[\vec{R}\,\vec{\alpha}\,\vec{\beta}](\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2}$

Sol. $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\alpha}\times\vec{\beta}$ are three non-coplanar vectors. Any vector $\vec{R}$ can be represented as a linear combination of these vectors. Thus,

$$\vec{R} = k_1\vec{\alpha} + k_2\vec{\beta} + k_3(\vec{\alpha}\times\vec{\beta})$$

Take dot product of (i) with $(\vec{\alpha}\times\vec{\beta})$. We have

$$\vec{R}\cdot(\vec{\alpha}\times\vec{\beta}) = k_3(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\beta}) = k_3|\vec{\alpha}\times\vec{\beta}|^2$$

or $$k_3 = \frac{\vec{R}\cdot(\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2} = \frac{[\vec{R}\,\vec{\alpha}\,\vec{\beta}]}{|\vec{\alpha}\times\vec{\beta}|^2}$$

Take dot product of (i) with $\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta})$. We have

$$\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta})) = k_2(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))\cdot\vec{\beta}$$

$$= k_2\,[(\vec{\alpha}\cdot\vec{\beta})\vec{\alpha} - (\vec{\alpha}\cdot\vec{\alpha})\vec{\beta}]\cdot\vec{\beta} = k_2[(\vec{\alpha}\cdot\vec{\beta})^2 - |\vec{\alpha}|^2|\beta|^2]$$

$$= -k_2|\vec{\alpha}\times\vec{\beta}|^2$$

or $$k_2 = \frac{-[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]}{|\vec{\alpha}\times\vec{\beta}|^2} \qquad \text{Similarly, } k_1 = -\frac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]}{|\vec{\alpha}\times\vec{\beta}|^2}$$

$\Rightarrow$ $$\vec{R} = \frac{-[\vec{R}\cdot[\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\,\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2} - \frac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\,\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2} + \frac{[(\vec{R}\cdot(\vec{\alpha}\times\vec{\beta}))]\,(\vec{\alpha}\times\vec{\beta})}{(\vec{\alpha}\times\vec{\beta})^2}$$

or $$\vec{R} + \frac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2} + \frac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\cdot\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2} = \frac{[\vec{R}\cdot(\vec{\alpha}\times\vec{\beta})]\,(\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2}$$

ILLUSTRATION 3.90

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar non-zero vectors, then prove that $(\vec{a}\cdot\vec{a})\ \vec{b}\times\vec{c} + (\vec{a}\cdot\vec{b})\ \vec{c}\times\vec{a} + (\vec{a}\cdot\vec{c})\ \vec{a}\times\vec{b} = [\vec{b}\ \vec{c}\ \vec{a}]\ \vec{a}$.

Sol. As $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, $\vec{b}\times\vec{a}$, $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ are also non-coplanar.

So, any vector can be expressed as a linear combination of these vectors.

Let $\vec{a} = \lambda\ \vec{b}\times\vec{c} + \mu\ \vec{c}\times\vec{a} + \nu\ \vec{a}\times\vec{b}$

$\therefore$ $\vec{a}\cdot\vec{a} = \lambda[\vec{b}\ \vec{c}\ \vec{a}],\ \vec{a}\cdot\vec{b} = \mu[\vec{c}\ \vec{a}\ \vec{b}],\ \vec{a}\cdot\vec{c} = \nu\ [\vec{a}\ \vec{b}\ \vec{c}]$

$\therefore$ $$\vec{a} = \frac{(\vec{a}\cdot\vec{a})\,\vec{b}\times\vec{c}}{[\vec{b}\ \vec{c}\ \vec{a}]} + \frac{(\vec{a}\cdot\vec{b})\,\vec{c}\times\vec{a}}{[\vec{c}\ \vec{a}\ \vec{b}]} + \frac{(\vec{a}\cdot\vec{c})\,\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

RECIPROCAL SYSTEM OF VECTORS

Two systems of vectors are called reciprocal systems of vectors if by taking the dot product we get unity.

Thus, if $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, and if

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} \text{ and } \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$, then $\vec{a}', \vec{b}', \vec{c}'$ are said to be the reciprocal systems of vectors for vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

PROPERTIES

I. If $\vec{a}, \vec{b}$ and $\vec{c}$ and $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are reciprocal system of vectors, then $\vec{a}\cdot\vec{a}' = \dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{(\vec{a}\ \vec{b}\ \vec{c})} = \dfrac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 1$.

Similarly, $\vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$.

Due to the above property, the two systems of vectors are called reciprocal systems.

II. $\vec{a}\cdot\vec{b}' = \vec{a}\cdot\vec{c}' = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = \vec{c}\cdot\vec{b}' = 0$

III. $[\vec{a}\ \vec{b}\ \vec{c}][\vec{a}'\vec{b}'\vec{c}'] = 1$

Proof:

We have $$[\vec{a}'\vec{b}'\vec{c}'] = \left[\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}\ \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}\ \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}\right] = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}[\vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\ \vec{a}\times\vec{b}]$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}[\vec{a}\ \vec{b}\ \vec{c}]^2 = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$\Rightarrow$ $$[\vec{a}'\vec{b}'\vec{c}'][\vec{a}\ \vec{b}\ \vec{c}] = 1$$

IV. The orthogonal triad of vectors $\hat{i}, \hat{j}$ and $\hat{k}$ is self-reciprocal.

Let $\hat{i}', \hat{j}'$ and $\hat{k}'$ be the system of vectors reciprocal to the system $\hat{i}, \hat{j}$ and $\hat{k}$. Then, we have

$$\hat{i}' = \frac{\hat{j} \times \hat{k}}{[\hat{i}\,\hat{j}\,\hat{k}]} = \hat{i}.$$

Similarly, $\hat{j}' = \hat{j}$ and $\hat{k}' = \hat{k}$.

V. $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar iff $\vec{a}', \vec{b}'$ and $\vec{c}'$ are non-coplanar.

As $[\vec{a}\,\vec{b}\,\vec{c}] \cdot [\vec{a}'\,\vec{b}'\,\vec{c}'] = 1$ and $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$ are non-coplanar $\Leftrightarrow \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]} \neq 0 \Leftrightarrow [\vec{a}'\,\vec{b}'\,\vec{c}']$ are non-coplanar.

ILLUSTRATION 3.91

Find a set of vectors reciprocal to the set $-\hat{i}+\hat{j}+\hat{k}, \hat{i}-\hat{j}+\hat{k}, \hat{i}+\hat{j}+\hat{k}$.

Sol. Let $\vec{a} = -\hat{i}+\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-\hat{j}+\hat{k}$, $\vec{c} = \hat{i}+\hat{j}+\hat{k}$

Then
$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2\hat{i}+2\hat{k}, \quad \vec{c} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -2\hat{j}+2\hat{k},$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\hat{i}+2\hat{j}$$

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 4$$

If a', b', c' is the reciprocal system of vectors, then

$$\vec{a}' = (\vec{b} \times \vec{c})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(-\hat{i}+\hat{k}), \quad \vec{b}' = (\vec{c} \times \vec{a})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(-\hat{j}+\hat{k}),$$

$$\vec{c}' = (\vec{a} \times \vec{b})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(\hat{i}+\hat{j})$$

ILLUSTRATION 3.92

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be a set of non-coplanar vectors and $\vec{a}', \vec{b}'$ and $\vec{c}'$ be its reciprocal set. Prove that

$$\vec{a} = \frac{\vec{b}' \times \vec{c}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}, \quad \vec{b} = \frac{\vec{c}' \times \vec{a}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']} \text{ and } \vec{c} = \frac{\vec{a}' \times \vec{b}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}.$$

Sol. We have, $\vec{b'}\times\vec{c'} = \dfrac{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$

$$= \frac{\{(\vec{c}\times\vec{a})\cdot\vec{b}\}\vec{a}-\{(\vec{c}\times\vec{a})\cdot\vec{a}\}\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{[\vec{c}\,\vec{a}\,\vec{b}]\vec{a}-[\vec{c}\,\vec{a}\,\vec{a}]\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]\vec{a}-0}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Also, $[\vec{a'}\,\vec{b'}\,\vec{c'}] = \vec{a'}\cdot(\vec{b'}\times\vec{c'}) = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\cdot\dfrac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$\Rightarrow$ $\dfrac{\vec{b'}\times\vec{c'}}{[\vec{a'}\,\vec{b'}\,\vec{c'}]} = \vec{a}$

Similarly, $\vec{b} = \dfrac{\vec{c'}\times\vec{a'}}{[\vec{a'}\,\vec{b'}\,\vec{c'}]}$, $\vec{c} = \dfrac{\vec{a'}\times\vec{b'}}{[\vec{a'}\,\vec{b'}\,\vec{c'}]}$

ILLUSTRATION 3.93

If $\vec{a},\vec{b},\vec{c}$ and $\vec{a'},\vec{b'},\vec{c'}$ are reciprocal system of vectors, then prove that $\vec{a'}\times\vec{b'}+\vec{b'}\times\vec{c'}+\vec{c'}\times\vec{a'} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$.

Sol. $\vec{a'}\times\vec{b'} = \dfrac{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \dfrac{\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c}-\{(\vec{b}\times\vec{c})\cdot\vec{c}\}\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \dfrac{[\vec{b}\,\vec{c}\,\vec{a}]\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \dfrac{\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Similarly, $\vec{b'}\times\vec{c'} = \dfrac{\vec{a}}{[\vec{a}\times\vec{b}\times\vec{c}]}$ and $\vec{c'}\times\vec{a'} = \dfrac{\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Adding, $\vec{a'}\times\vec{b'}+\vec{b'}\times\vec{c'}+\vec{c'}\times\vec{a'} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

ILLUSTRATION 3.94

If $\vec{a},\vec{b}$ and $\vec{c}$ be three non-coplanar vectors and a', b' and c' constitute the reciprocal system of vectors, then prove that

i. $\vec{r} = (\vec{r}\cdot\vec{a'})\vec{a}+(\vec{r}\cdot\vec{b'})\vec{b}+(\vec{r}\cdot\vec{c'})\vec{c}$

ii. $\vec{r} = (\vec{r}\cdot\vec{a})\vec{a'}+(\vec{r}\cdot\vec{b})\vec{b'}+(\vec{r}\cdot\vec{c})\vec{c'}$

Sol.

i. Since a vector can be expressed as a linear combination of three non-coplanar vectors, let

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$$

where x, y and z are scalars. (i)

Multiplying both sides of (i) scalarly by $\vec{a'}$, we get

$$\vec{r}\cdot\vec{a'} = x\vec{a}\cdot\vec{a'} + y\vec{b}\cdot\vec{a'} + z\vec{c}\cdot\vec{a'} = x\cdot 1 = x \qquad (\because \vec{a}\cdot\vec{a'} = 1,\ \vec{b}\cdot\vec{a'} = 0 = \vec{c}\cdot\vec{a'})$$

Similarly, multiplying both sides of (i) scalarly by $\vec{b'}$ and $\vec{c'}$, successively, we get

$$y = \vec{r}\cdot\vec{b'} \text{ and } z = \vec{r}\cdot\vec{c'}$$

Putting in (i), we get $\vec{r} = (\vec{r}\cdot\vec{a'})\vec{a} + (\vec{r}\cdot\vec{b'})\vec{b} + (\vec{r}\cdot\vec{c'})\vec{c}$

ii. Since $\vec{a'}$, $\vec{b'}$ and $\vec{c'}$ are three non-coplanar vectors, we can take $\vec{r} = x\vec{a'} + y\vec{b'} + z\vec{c'}$ (ii)

Multiplying both sides of (ii) scalarly by $\vec{a}$, we get

$$\vec{r}\cdot\vec{a} = x(\vec{a'}\cdot\vec{a}) + y(\vec{b'}\cdot\vec{a}) + z(\vec{c'}\cdot\vec{a}) = x \qquad (\because \vec{a'}\cdot\vec{a} = 1,\ \vec{b'}\cdot\vec{a} = 0 = \vec{c'}\cdot\vec{a})$$

Similarly, multiplying both sides of (i) scalarly by $\vec{b}$ and $\vec{c}$ successively, we get

$$y = \vec{r}\cdot\vec{b} \text{ and } z = \vec{r}\cdot\vec{c}$$

Putting in (ii), we get $\vec{r} = (\vec{r}\cdot\vec{a})\vec{a'} + (\vec{r}\cdot\vec{b})\vec{b'} + (\vec{r}\cdot\vec{c})\vec{c'}$.

CONCEPT APPLICATION EXERCISE 3.3

1. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four non-coplanar unit vectors such that $\vec{d}$ makes equal angles with all the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, then prove that $[\vec{d}\ \vec{a}\ \vec{b}] = [\vec{d}\ \vec{c}\ \vec{b}] = [\vec{d}\ \vec{c}\ \vec{a}]$.

2. Prove that if $[\vec{l}\ \vec{m}\ \vec{n}]$ are three non-coplanar vectors, then $[\vec{l}\ \vec{m}\ \vec{n}](\vec{a}\times\vec{b}) = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}$

3. If the volume of a parallelepiped whose adjacent edges are $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + \alpha\hat{j} + 2\hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} + \alpha\hat{k}$ is 15, then find the value of α if $(\alpha > 0)$.

4. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$, then find vector $\vec{c}$ such that $\vec{a}\cdot\vec{c} = 2$ and $\vec{a}\times\vec{c} = \vec{b}$.

5. If $\vec{x}\cdot\vec{a} = 0$, $\vec{x}\cdot\vec{b} = 0$ and $\vec{x}\cdot\vec{c} = 0$ for some non-zero vector $\vec{x}$, then prove that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

6. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a} \times \vec{b} = \vec{c}, \vec{b} \times \vec{c} = \vec{a}, \vec{c} \times \vec{a} = \vec{b}$, then prove that $|\vec{a}| = |\vec{b}| = |\vec{c}|$.

7. If $\vec{a} = \vec{p} + \vec{q}, \vec{p} \times \vec{b} = \vec{0}$ and $\vec{q} \cdot \vec{b} = 0$, then prove that $\frac{\vec{b} \times (\vec{a} \times \vec{b})}{\vec{b} \cdot \vec{b}} = \vec{q}$.

8. Prove that $(\vec{a} \cdot (\vec{b} \times \hat{i})\,\hat{i} + (\vec{a} \cdot (\vec{b} \times \hat{j}))\,\hat{j} + (\vec{a} \cdot (\vec{b} \times \hat{k}))\,\hat{k} = \vec{a} \times \vec{b}$.

9. For any four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, prove that $\vec{d} \cdot (\vec{a} \times (\vec{b} \times (\vec{c} \times \vec{d}))) = (\vec{b} \cdot \vec{d})[\vec{a}\ \vec{c}\ \vec{d}]$.

10. If $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors such that $\vec{a} \times (\vec{a} \times \vec{b}) = \frac{1}{2}\vec{b}$, then find the angle between $\vec{a}$ and $\vec{b}$.

11. Show that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ if and only if $\vec{a}$ and $\vec{c}$ are collinear or $(\vec{a} \times \vec{c}) \times \vec{b} = \vec{0}$.

12. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-zero vectors such that no two are collinear and $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$. If θ is the acute angle between vectors $\vec{b}$ and $\vec{c}$, then find the value of $\sin\theta$.

13. If $\vec{p}, \vec{q}, \vec{r}$ denote vectors $\vec{b} \times \vec{c}, \vec{c} \times \vec{a}, \vec{a} \times \vec{b}$, respectively, show that $\vec{a}$ is parallel to $\vec{q} \times \vec{r}, \vec{b}$ is parallel to $\vec{r} \times \vec{p}, \vec{c}$ is parallel to $\vec{p} \times \vec{q}$.

14. Let $\vec{a}, \vec{b}, \vec{c}$ be non-coplanar vectors and let the equations $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vector $\vec{a}, \vec{b}, \vec{c}$, then prove that $\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}'$ is a null vector.

15. Given unit vectors $\hat{m}, \hat{n}$ and $\hat{p}$ such that angle between $\hat{m}$ and $\hat{n}$ is α and angle between $\hat{p}$ and $(\hat{m} \times \hat{n})$ is also α, if $[\hat{n}\ \hat{p}\ \hat{m}] = 1/4$, then find the value of α.

16. $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors and every two are inclined to each other at an angle $\cos^{-1}(3/5)$. If $\vec{a} \times \vec{b} = p\vec{a} + q\vec{b} + r\vec{c}$, where p, q, r are scalars, then find the value of q.

17. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then prove that $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = \frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

ANSWERS

3. $\alpha = -1, 9/2$

4. $\vec{c} = \frac{1}{3}(-\hat{i} + 2\hat{j} + 5\hat{k})$

10. Information is inconsistent.

12. $\sin\theta = \frac{2\sqrt{2}}{3}$

15. $\alpha = 15°$

16. $q = -\frac{3}{\sqrt{55}}$

Solved Examples

EXAMPLE 3.1

From a point O inside a triangle ABC, perpendiculars OD, OE and OF are drawn to the sides BC, CA and AB, respectively. Prove that the perpendiculars from A, B and C to the sides EF, FD and DE are concurrent.

Sol. Let the position vectors of points A, B, C, D, E and F be $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$ and $\vec{f}$ w.r.t. O. Let perpendiculars from A to EF and from B to DF meet each other at H. Let position vectors of H be $\vec{r}$. We join CH. In order to prove the statement given in the question, it is sufficient to prove that CH is perpendicular to DE.

Now, as $OD \perp BC \Rightarrow \vec{d}\cdot(\vec{b}-\vec{c}) = 0$

$$\Rightarrow \quad \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} \qquad \text{(i)}$$

as $OE \perp AC \Rightarrow \vec{e}\cdot(\vec{c}-\vec{a}) = 0 \Rightarrow \vec{e}\cdot\vec{c} = \vec{e}\cdot\vec{a}$ (ii)

as $OF \perp AB \Rightarrow \vec{f}\cdot(\vec{a}-\vec{b}) = 0 \Rightarrow \vec{f}\cdot\vec{a} = \vec{f}\cdot\vec{b}$ (iii)

Also $\quad AH \perp EF \Rightarrow (\vec{r}-\vec{a})\cdot(\vec{e}-\vec{f}) = 0$

$$\Rightarrow \quad \vec{r}\cdot\vec{e} - \vec{r}\cdot\vec{f} - \vec{a}\cdot\vec{e} + \vec{a}\cdot\vec{f} = 0 \qquad \text{(iv)}$$

and $\quad BH \perp FD \Rightarrow (\vec{r}-\vec{b})\cdot(\vec{f}-\vec{d}) = 0$

$$\Rightarrow \quad \vec{r}\cdot\vec{f} - \vec{r}\cdot\vec{d} - \vec{b}\cdot\vec{f} + \vec{b}\cdot\vec{d} = 0 \qquad \text{(v)}$$

Adding (iv) and (v), we get

$$\vec{r}\cdot\vec{e} - \vec{a}\cdot\vec{e} + \vec{a}\cdot\vec{f} - \vec{r}\cdot\vec{d} - \vec{b}\cdot\vec{f} + \vec{b}\cdot\vec{d} = 0$$

or $\quad \vec{r}\cdot(\vec{e}-\vec{d}) - \vec{e}\cdot\vec{c} + \vec{d}\cdot\vec{c} = 0$ [using (i), (ii) and (iii))]

or $\quad (\vec{r}-\vec{c})\cdot(\vec{e}-\vec{d}) = 0$

$$\Rightarrow \quad \overrightarrow{CH}\cdot\overrightarrow{ED} = 0 \Rightarrow CH \perp ED$$

EXAMPLE 3.2

Determine the value of c so that for all real x, vectors $cx\,\hat{i} - 6\hat{j} - 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\,\hat{k}$ make an obtuse angle with each other.

Sol. We have, $\vec{a} = cx\,\hat{i} - 6\hat{j} - 3\hat{k}$

$$\vec{b} = x\hat{i} + 2\hat{j} + 2cx\,\hat{k}$$

Now we know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

As the angle between $\vec{a}$ and $\vec{b}$ is obtuse,

$\cos\theta < 0$

$\vec{a}\cdot\vec{b} < 0$

$\Rightarrow \quad cx^2 - 12 + 6cx < 0$

$\Rightarrow \quad -cx^2 - 6cx + 12 > 0,\ x \in R$

$\Rightarrow \quad -c > 0$ and $D < 0$

$\Rightarrow \quad c < 0$ and $36c^2 + 48c < 0$

$\Rightarrow \quad c < 0$ and $(3c + 4) > 0$

$\Rightarrow \quad c < 0$ and $c > -4/3$

$\Rightarrow \quad -4/3 < c < 0$

EXAMPLE 3.3

For any two vectors $\vec{u}$ and $\vec{v}$, prove that

a. $(\vec{u}\cdot\vec{v})^2 + |\vec{u}\times\vec{v}|^2 = |\vec{u}|^2\,|\vec{v}|^2$ and

b. $(\vec{1} + |\vec{u}|^2)(\vec{1} + |\vec{v}|^2) = (1 - \vec{u}\cdot\vec{v})^2 + |\vec{u} + \vec{v} + (\vec{u}\times\vec{v})|^2$

Sol.

a. We have $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

and $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\,\hat{n}$

(where θ is the angle between $\vec{u}$ and $\vec{v}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$)

$\Rightarrow \quad (\vec{u}\cdot\vec{v})^2 + |\vec{u}\times\vec{v}|^2$

$= |\vec{u}|^2\,|\vec{v}|^2(\cos^2\theta + \sin^2\theta) = |\vec{u}|^2\,|\vec{v}|^2$

b. $(1 - \vec{u}\cdot\vec{v})^2 + |\vec{u} + \vec{v} + (\vec{u}\times\vec{v})|^2$

$= 1 - 2\vec{u}\cdot\vec{v} + (\vec{u}\cdot\vec{v})^2 + |\vec{u}|^2 + |\vec{v}|^2 + |\vec{u}\times\vec{v}|^2 + 2\vec{u}\cdot\vec{v}\quad (\because \vec{u}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{u}\times\vec{v}) = 0)$

$= 1 + |\vec{u}|^2 + |\vec{v}|^2 + (\vec{u}\cdot\vec{v})^2 + |\vec{u}\times\vec{v}|^2$

$= 1 + |\vec{u}|^2 + |\vec{v}|^2 + |\vec{u}|^2\,|\vec{v}|^2$

$= (\vec{1} + |\vec{u}|^2)(\vec{1} + |\vec{v}|^2)$

EXAMPLE 3.4

If vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, show that $\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = \vec{0}$.

Sol. Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three coplanar vectors. Therefore, there exist scalars x, y and z, not all zero, such that

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \quad \text{(i)}$$

Taking dot product of $\vec{a}$ and (i), we get

$$x\vec{a}\cdot\vec{a} + y\vec{a}\cdot\vec{b} + z\vec{a}\cdot\vec{c} = 0 \quad \text{(ii)}$$

Again taking dot product of $\vec{b}$ and (i), we get

$$x\vec{b}\cdot\vec{a} + y\vec{b}\cdot\vec{b} + z\vec{b}\cdot\vec{c} = 0 \quad \text{(iii)}$$

Now Eqs. (i), (ii) and (iii) form a homogeneous system of equations, where x, y and z are not all zero. Therefore, the system must have a non-trivial solution, and for this, the determinant of coefficient matrix should be zero, i.e.,

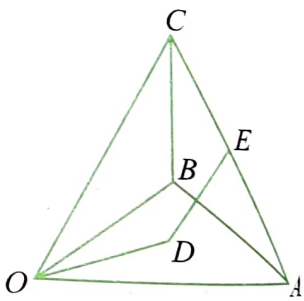
$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = 0$$

EXAMPLE 3.5

OABC is a regular tetrahedron in which *D* is the circumcentre of ΔOAB and *E* is the midpoint of edge *AC*. Prove that *DE* is equal to half the edge of tetrahedron.

Sol. Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Since ΔABC is equilateral, the circumcenters of triangle coincide with the centroid. Therefore,



$$\overrightarrow{OD} = \frac{\vec{a}+\vec{b}}{3}$$

Also $$\overrightarrow{OE} = \frac{\vec{a}+\vec{c}}{2}$$

$\therefore$ $$\overrightarrow{DE} = \frac{\vec{a}+3\vec{c}-2\vec{b}}{6}$$

$\therefore$ $$|\overrightarrow{DE}|^2 = \left|\frac{\vec{a}+3\vec{c}-2\vec{b}}{6}\right|^2$$

Since $|\vec{a}| = |\vec{b}| = |\vec{c}|$ and $\theta = 60°$, we have

$$|\overrightarrow{DE}|^2 = \frac{1}{36}\left(a^2 + 9a^2 + 4a^2 + \frac{6a^2}{2} - \frac{12a^2}{2} - \frac{4a^2}{2}\right)$$

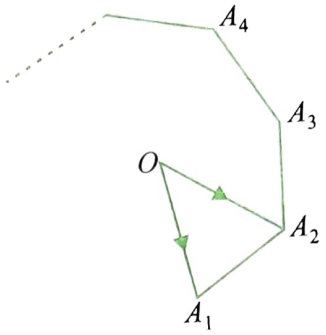
$$= \frac{a^2}{4}$$

or $$|\overrightarrow{DE}| = \frac{a}{2}$$

EXAMPLE 3.6

$A_1, A_2, \ldots, A_n$ are the vertices of a regular plane polygon with n sides and O as its centre. Show that $\sum_{i=1}^{n-1} (\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = (1-n)(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$.

Sol. $\overrightarrow{OA_1}\ \overrightarrow{OA_2}, \ldots, \overrightarrow{OA_n}$. All vectors are of same magnitude, say a, and angle between any two consecutive vectors is the same, that is, $2\pi/n$. Let $\hat{p}$ be the unit vector parallel to the plane of the polygon.



Let $\overrightarrow{OA_1} \times \overrightarrow{OA_2} = a^2 \sin\frac{2\pi}{n}\hat{p}$ (i)

Now, $\sum_{i=1}^{n-1} \overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}} = \sum_{i=1}^{n-1} a^2 \sin\frac{2\pi}{n}\hat{p}$

$= (n-1)\, a^2 \sin\frac{2\pi}{n}\hat{p}$

$= (n-1)\,[-\overrightarrow{OA_2} \times \overrightarrow{OA_1}]$ [Using (i)]

$= (1-n)\,[\overrightarrow{OA_2} \times \overrightarrow{OA_1}]$

= R.H.S.

EXAMPLE 3.7

Let $\vec{A} = 2\hat{i} + \hat{k}, \vec{B} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$. Determine a vector $\vec{R}$ satisfying $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R} \cdot \vec{A} = 0$.

Sol. We are given that $\vec{A} = 2\hat{i} + \hat{k}, \vec{B} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ and to determine a vector $\vec{R}$ such that $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R} \cdot \vec{A} = 0$.

Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$.

Then $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$

$$\Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 7 \\ 1 & 1 & 1 \end{vmatrix}$$

or $(y-z)\hat{i} - (x-z)\hat{j} + (x-y)\hat{k} = -10\hat{i} + (x-z)\hat{j} + 7\hat{k}$

$y - z = -10$ (i)

$x - z = -3$ (ii)

$x - y = 7$ (iii)

Also $\vec{R} \cdot \vec{A} = 0$

$\Rightarrow \quad 2x + z = 0$ (iv)

Substituting $y = x - 7$ and $z = -2x$ from (iii) and (iv), respectively in (i), we get

$$x - 7 + 2x = -10$$

$$\Rightarrow \quad 3x = -3$$

$$\Rightarrow \quad x = -1, y = -8 \text{ and } z = 2$$

EXAMPLE 3.8

Let O be an interior point of ΔABC such that $\overrightarrow{OA} + 2\overrightarrow{OB} + 3\overrightarrow{OC} = \vec{0}$. Then find the ratio of the area of ΔABC to the area of ΔAOC.

Sol. $$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta AOC} = \frac{\frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{\frac{1}{2}|\vec{a} \times \vec{c}|}$$

Now $$\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$$

Cross multiply with $\vec{b}$, we get

$$\vec{a} \times \vec{b} + 3\vec{c} \times \vec{b} = \vec{0}$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = (\vec{b} \times \vec{c})$$

Cross multiply with $\vec{a}$, we get

$$2\vec{a} \times \vec{b} + 3\vec{a} \times \vec{c} = 0$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = \frac{3}{2}(\vec{c} \times \vec{a})$$

$$\therefore \quad \vec{a} \times \vec{b} = \frac{3}{2}(\vec{c} \times \vec{a}) = 3(\vec{b} \times \vec{c})$$

Let $(\vec{c} \times \vec{a}) = \vec{p}$; $\vec{a} \times \vec{b} = \frac{3\vec{p}}{2}$; $\vec{b} \times \vec{c} = \frac{\vec{p}}{2}$

$$\therefore \quad \text{Ratio} = \frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{|\vec{c} \times \vec{a}|} = \frac{\left|\frac{3\vec{p}}{2} + \frac{\vec{p}}{2} + \vec{p}\right|}{|\vec{p}|} = \frac{3|\vec{p}|}{|\vec{p}|} = 3$$

EXAMPLE 3.9

Given that vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a triangle such that $\vec{A} = \vec{B} + \vec{C}$. Find a, b, c and d such that the area of the triangle is $5\sqrt{6}$ where

$$\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{B} = d\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{C} = 3\hat{i} + \hat{j} - 2\hat{k}$$

Sol. Here $\vec{A}$, $\vec{B}$ and $\vec{C}$ are the vectors representing the sides of triangle ABC, where $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$, $\vec{B} = \hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{C} = 3\hat{i} + \hat{j} - 2\hat{k}$.

Given that $\vec{A} = \vec{B} + \vec{C}$. Therefore

$$a\hat{i} + b\hat{j} + c\hat{k} = (d + 3)\hat{i} + 4\hat{j} + 2\hat{k}$$

$$a = d + 3,\ b = 4,\ c = 2$$

$\Rightarrow$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d & 3 & 4 \\ 3 & 1 & -2 \end{vmatrix}$$

Now

$$= -10\hat{i} + (2d + 12)\hat{j} + (d - 9)\hat{k}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|\vec{B} \times \vec{C}|$$

$\therefore$

$$= \frac{1}{2}\sqrt{[100 + (2d + 12)^2 + (d - 9)^2]}$$

$$= 5\sqrt{6} \qquad \text{(Given)}$$

$$\sqrt{(5d^2 + 30d + 325)} = 10\sqrt{6}$$

or $5d^2 + 30d - 275 = 0$ or $d^2 + 6d - 55 = 0$

or $(d + 11)(d - 5) = 0$

or

$\Rightarrow$ $d = 5$ or -11

When $d = 5$, $a = 8$, $b = 4$ and $c = 2$, and when $d = -11$, $a = -8$, $b = 4$ and $c = 2$.

EXAMPLE 3.10

A line l is passing through the point $\vec{b}$ and is parallel to vector $\vec{c}$. Determine the distance of point $A(\vec{a})$ from the line l in the form $\left|\vec{b} - \vec{a} + \frac{(\vec{a} - \vec{b})\cdot\vec{c}}{|\vec{c}|^2}\vec{c}\right|$ or $\frac{|(\vec{b} - \vec{a}) \times \vec{c}|}{|\vec{c}|}$.

Sol. $AM = |\overrightarrow{AB} \sin\theta|$, where θ is the angle between $\overrightarrow{AB}$ and $\vec{c}$ and $\sin\theta = \frac{|\overrightarrow{AB} \times \vec{c}|}{|\overrightarrow{AB}||\vec{c}|}$.

$\Rightarrow$

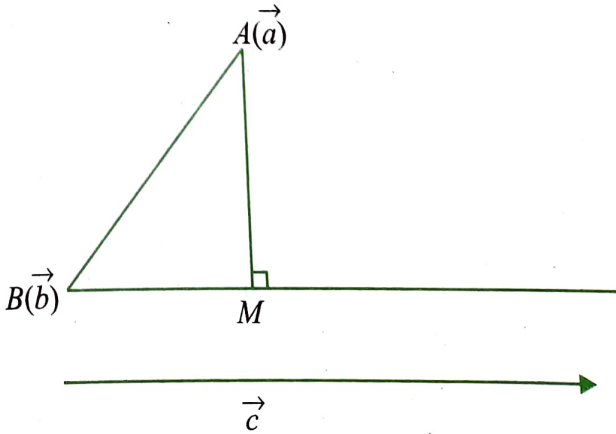
$$AM = |\overrightarrow{AB}| \frac{|\overrightarrow{AB} \times \vec{c}|}{|\overrightarrow{AB}||\vec{c}|} = \frac{|(\vec{b} - \vec{a}) \times \vec{c}|}{|\vec{c}|}$$

Also

$$\overrightarrow{BM} = \frac{(\vec{a} - \vec{b})\cdot\vec{c}}{|\vec{c}|}\frac{\vec{c}}{|\vec{c}|}$$

And

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$$



$$\Rightarrow \quad |\overrightarrow{AM}| = b\left|\vec{b} - \vec{a} + \frac{(\vec{a} - \vec{b})\,\vec{c}}{|\vec{c}|^2}\,\vec{c}\right|$$

EXAMPLE 3.11

If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{E}_1, \vec{E}_2, \vec{E}_3$ are two sets of vectors such that $\vec{e}_i \cdot \vec{E}_j = 1$, if $i = j$ and $\vec{e}_i \cdot \vec{E}_j = 0$ and if $i \neq j$, then prove that $[\vec{e}_1\ \vec{e}_2\ \vec{e}_3][\vec{E}_1\ \vec{E}_2\ \vec{E}_3] = 1$.

Sol. We know that

$$[\vec{e}_1\ \vec{e}_2\ \vec{e}_3][\vec{E}_1\ \vec{E}_2\ \vec{E}_3] = \begin{vmatrix} \vec{e}_1 \cdot \vec{E}_1 & \vec{e}_1 \cdot \vec{E}_2 & \vec{e}_1 \cdot \vec{E}_3 \\ \vec{e}_2 \cdot \vec{E}_1 & \vec{e}_2 \cdot \vec{E}_2 & \vec{e}_2 \cdot \vec{E}_2 \\ \vec{e}_3 \cdot \vec{E}_1 & \vec{e}_3 \cdot \vec{E}_2 & \vec{e}_3 \cdot \vec{E}_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

EXAMPLE 3.12

If $\vec{a}, \vec{b}, \vec{c}$ are three given non-coplanar vectors and any arbitrary vector $\vec{r}$ in space, where

$$\Delta_1 = \begin{vmatrix} \vec{r} \cdot \vec{a} & \vec{b} \cdot \vec{a} & \vec{c} \cdot \vec{a} \\ \vec{r} \cdot \vec{b} & \vec{b} \cdot \vec{b} & \vec{c} \cdot \vec{b} \\ \vec{r} \cdot \vec{c} & \vec{b} \cdot \vec{c} & \vec{c} \cdot \vec{c} \end{vmatrix}, \Delta_2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{r} \cdot \vec{a} & \vec{c} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{r} \cdot \vec{b} & \vec{c} \cdot \vec{b} \\ \vec{a} \cdot \vec{c} & \vec{r} \cdot \vec{c} & \vec{c} \cdot \vec{c} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{b} \cdot \vec{a} & \vec{r} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} & \vec{r} \cdot \vec{b} \\ \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} & \vec{r} \cdot \vec{c} \end{vmatrix}, \Delta = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{b} \cdot \vec{a} & \vec{c} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} & \vec{c} \cdot \vec{b} \\ \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} & \vec{c} \cdot \vec{c} \end{vmatrix}$$, then prove that $\vec{r} = \frac{\Delta_1}{\Delta}\vec{a} + \frac{\Delta_2}{\Delta}\vec{b} + \frac{\Delta_3}{\Delta}\vec{c}$.

Sol.

$$\Delta_1 = \begin{vmatrix} \vec{r} \cdot \vec{a} & \vec{b} \cdot \vec{a} & \vec{c} \cdot \vec{a} \\ \vec{r} \cdot \vec{b} & \vec{b} \cdot \vec{b} & \vec{c} \cdot \vec{b} \\ \vec{r} \cdot \vec{c} & \vec{b} \cdot \vec{c} & \vec{c} \cdot \vec{c} \end{vmatrix} = [\vec{r}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Delta_2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{r} \cdot \vec{a} & \vec{c} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{r} \cdot \vec{b} & \vec{c} \cdot \vec{b} \\ \vec{a} \cdot \vec{c} & \vec{r} \cdot \vec{c} & \vec{c} \cdot \vec{c} \end{vmatrix} = [\vec{r}\ \vec{c}\ \vec{a}][\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Delta_3 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{b} \cdot \vec{a} & \vec{r} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} & \vec{r} \cdot \vec{b} \\ \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} & \vec{r} \cdot \vec{c} \end{vmatrix} = [\vec{r}\ \vec{a}\ \vec{b}][\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Delta = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{b}\cdot\vec{a} & \vec{c}\cdot\vec{a} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} & \vec{c}\cdot\vec{b} \\ \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c} \end{vmatrix} = [\vec{a}\,\vec{b}\,\vec{c}]^2$$

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$$

Now

Taking dot product with $\vec{a}\times\vec{b}$, we have

$$\vec{r}\cdot(\vec{a}\times\vec{b}) = z\vec{c}\cdot(\vec{a}\times\vec{b})$$

$$\Rightarrow \quad z = \frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{\Delta_3}{\Delta}$$

Similarly $x = \dfrac{\Delta_1}{\Delta}$ and $y = \dfrac{\Delta_2}{\Delta}$

$$\Rightarrow \quad \vec{r} = \frac{\Delta_1}{\Delta}\vec{a} + \frac{\Delta_2}{\Delta}\vec{b} + \frac{\Delta_3}{\Delta}\vec{c}$$

EXAMPLE 3.13

If c is a given non-zero scalar, and $\vec{A}$ and $\vec{B}$ are given non-zero vectors such that $\vec{A}\perp\vec{B}$, then find vector $\vec{X}$ which satisfies the equations $\vec{A}\cdot\vec{X} = c$ and $\vec{A}\times\vec{X} = \vec{B}$.

Sol.

$$\vec{A}\times\vec{X} = \vec{B}$$

or $$(\vec{A}\times\vec{X})\times\vec{A} = \vec{B}\times\vec{A}$$

or $$(\vec{A}\cdot\vec{A})\vec{X} - (\vec{X}\cdot\vec{A})\vec{A} = \vec{B}\times\vec{A}$$

or $$(\vec{A}\cdot\vec{A})\vec{X} - c\vec{A} = \vec{B}\times\vec{A}$$

or $$\vec{X} = \frac{\vec{B}\times\vec{A} + c\vec{A}}{(\vec{A}\cdot\vec{A})}$$

EXAMPLE 3.14

If vectors $\vec{b}, \vec{c}$ and $\vec{d}$ are not coplanar, then prove that vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to $\vec{a}$.

Sol.

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$

Here

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -(\vec{c}\times\vec{d}\cdot\vec{b})\vec{a} + (\vec{c}\times\vec{d}\cdot\vec{a})\vec{b} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a} \qquad \text{(i)}$$

$$(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) = -(\vec{d}\times\vec{b}\cdot\vec{c})\vec{a} + (\vec{d}\times\vec{b}\cdot\vec{a})\vec{c} = [\vec{a}\,\vec{d}\,\vec{b}]\vec{c} - [\vec{c}\,\vec{d}\,\vec{b}]\vec{a} \qquad \text{(ii)}$$

$$(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{d}\cdot\vec{c})\vec{b} - (\vec{a}\times\vec{d}\cdot\vec{b})\vec{c} = -[\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{a}\,\vec{d}\,\vec{b}]\vec{c} \qquad \text{(iii)}$$

(Note : Here we have tried to write the given expression in such a way that we can get terms involving $\vec{a}$ and other similar terms which can get cancelled)

Adding (i), (ii) and (iii), we get

Given vector $= -2\,[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a} = k\vec{a}$

Hence, given vector is parallel to $\vec{a}$.

EXAMPLE 3.15

Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$, then prove that $|(\vec{u} \times \vec{v}) \cdot \vec{w}| \leq 1/2$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.

Sol. $$[\vec{u}\ \vec{v}\ \vec{w}] = (\vec{u} \times \vec{v}) \cdot (\vec{v} - \vec{w} \times \vec{u}) = (\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{w}) = \begin{vmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{w} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{w} \end{vmatrix}$$

Now, $\vec{u} \cdot \vec{u} = 1$

$$\vec{u} \cdot \vec{w} = \vec{u} \cdot (\vec{v} - \vec{w} \times \vec{u}) = \vec{u} \cdot \vec{v} - [\vec{u}\ \vec{w}\ \vec{u}]$$

$$= \vec{u} \cdot \vec{v}$$

$$\vec{v} \cdot \vec{w} = \vec{v} \cdot (\vec{v} - \vec{w} \times \vec{u}) = 1 - [\vec{v}\ \vec{w}\ \vec{u}]$$

$$= 1 - [\vec{u}\ \vec{v}\ \vec{w}]$$

$$\therefore \quad [\vec{u}\ \vec{v}\ \vec{w}] = \begin{vmatrix} 1 & \cos\theta \\ \cos\theta & 1 - [\vec{u}\ \vec{v}\ \vec{w}] \end{vmatrix}$$ (θ is the angle between $\vec{u}$ and $\vec{v}$)

$$= 1 - [\vec{u}\ \vec{v}\ \vec{w}] - \cos^2\theta$$

$$= \frac{1}{2}\sin^2\theta \leq \frac{1}{2}$$

Equality holds when $\sin^2\theta = 1$, i.e., $\theta = \pi/2$, i.e., $\vec{u} \perp \vec{v}$.

EXAMPLE 3.16

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are distinct vectors such that $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$, prove that $(\vec{a} - \vec{d}) \cdot (\vec{b} - \vec{c}) \neq 0$, i.e., $\vec{a} \cdot \vec{b} + \vec{d} \cdot \vec{c} \neq \vec{d} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.

Sol. Given that $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ (i)

and $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ (ii)

Subtracting (ii) from (i), we get

$$\vec{a} \times (\vec{c} - \vec{b}) = (\vec{b} - \vec{c}) \times \vec{d}$$

$$\vec{a}\times(\vec{c}-\vec{b}) = \vec{d}\times(\vec{c}-\vec{b})$$

or $$\vec{a}\times(\vec{c}-\vec{b}) - \vec{d}\times(\vec{c}-\vec{b}) = 0$$

or $$(\vec{a}-\vec{d})\times(\vec{c}-\vec{d}) = 0$$

or $$(\vec{a}-\vec{d}) \,||\, (\vec{c}-\vec{d}) \quad (\because \vec{a}-\vec{d}\neq 0, \vec{c}-\vec{b}\neq 0)$$

or

Hence, the angle between $\vec{a}-\vec{d}$ and $\vec{c}-\vec{b}$ is either 0 or 180°.

$\Rightarrow$ $$(\vec{a}-\vec{d})\cdot(\vec{c}-\vec{b}) = |\vec{a}-\vec{d}||\vec{c}-\vec{b}| \cos 0 \neq 0$$

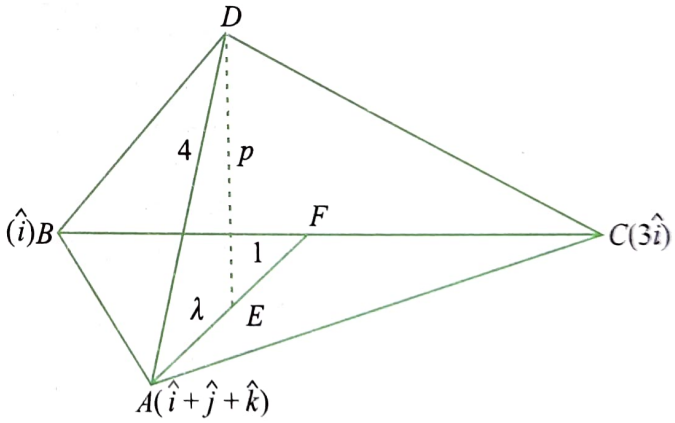
as $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ all are different.

EXAMPLE 3.17

The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}$ and $3\hat{i}$, respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $2\sqrt{2}/3$, find the position vectors of the point E for all its possible positions.

Sol. We are given $AD = 4$

Volume of tetrahedron $= \dfrac{2\sqrt{2}}{3}$



$\Rightarrow$ $$\frac{1}{3}(\text{Area of } \Delta ABC)\, p = \frac{2\sqrt{2}}{3}$$

$\therefore$ $$\frac{1}{2}|\overrightarrow{BA}\times\overrightarrow{BC}|\, p = 2\sqrt{2}$$

$$\frac{1}{2}\,|(\hat{j}+\hat{k})\times 2\hat{i}\,|\, p = 2\sqrt{2}$$

or $$|\hat{j}-\hat{k}|\, p = 2\sqrt{2}$$

or $$\sqrt{2}\, p = 2\sqrt{2} \text{ or } p = 2$$

We have to find the P.V. of point E. Let it divide median AF in the ratio $\lambda : 1$.

$$\text{P.V. of } E = \frac{\lambda\cdot 2\hat{i}+(\hat{i}+\hat{j}+\hat{k})}{\lambda+1}. \qquad \text{(i)}$$

$\therefore$ $$\overrightarrow{AE} = \text{P.V. or } E - \text{P.V. of } A = \frac{\lambda(\hat{i}-\hat{j}-\hat{k})}{\lambda+1}$$

$$|\overrightarrow{AE}|^2 = 3\left(\frac{\lambda}{\lambda+1}\right)^2 \qquad \text{(ii)}$$

Now, $$4+3\left(\frac{\lambda}{\lambda+1}\right)^2 = 16$$

$$\left(\frac{\lambda}{\lambda+1}\right)=\pm 2$$

$$\lambda=-2 \text{ or } -2/3$$

Putting the value of λ in (i), we get the P.V. of possible positions of E as $-\hat{i}+3\hat{j}+3\hat{k}$ or $3\hat{i}-\hat{j}-\hat{k}$.

EXAMPLE 3.18

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-coplanar unit vectors, equally inclined to one another at an angle θ. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$, find scalars p, q and r in terms of θ.

Sol. Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three unit vectors inclined at an angle θ with each other.

Also $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar. Therefore, $[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$.

Also given that $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$.

Taking dot product on both sides with $\vec{a}$, we get

$$p+q\cos\theta+r\cos\theta=[\vec{a}\ \vec{b}\ \vec{c}] \quad \text{(i)}$$

Similarly, taking dot product on both sides with $\vec{b}$ and $\vec{c}$, we get, respectively,

$$p\cos\theta+q+r\cos\theta=0 \quad \text{(ii)}$$

and

$$p\cos\theta+q\cos\theta+r=[\vec{a}\ \vec{b}\ \vec{c}] \quad \text{(iii)}$$

Adding (i), (ii) and (iii), we get

$$p+q+r=\frac{2[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta+1} \quad \text{(iv)}$$

Multiplying (iv) by $\cos\theta$ and subtracting (i) from it, we get

$$p(\cos\theta-1)=\frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta+1}-[\vec{a}\ \vec{b}\ \vec{c}]=\frac{-[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta+1}$$

or

$$p=\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{(1-\cos\theta)(1+2\cos\theta)}$$

Similarly, (iv) $\times\cos\theta$ – (ii) gives

$$q=\frac{-2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{(1+2\cos\theta)(1-\cos\theta)}$$

and (iv) $\times\cos\theta$ – (iii) gives

$$r(\cos\theta-1)=\frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta+1}-[\vec{a}\ \vec{b}\ \vec{c}]$$

or

$$r=\frac{-[\vec{a}\ \vec{b}\ \vec{c}]}{(2\cos\theta+1)(\cos\theta-1)}$$

But we have to find p, q and r in terms of θ only.

So let us find the value of $[\vec{a}\ \vec{b}\ \vec{c}]$

We know that

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+2\cos\theta & \cos\theta & \cos\theta \\ 1+2\cos\theta & 1 & \cos\theta \\ 1+2\cos\theta & \cos\theta & 1 \end{vmatrix} \qquad \text{(On operating } C_1 \to C_1 + C_2 + C_3\text{)}$$

$$= (1+2\cos\theta)\begin{vmatrix} 1 & \cos\theta & \cos\theta \\ 1 & 1 & \cos\theta \\ 1 & \cos\theta & 1 \end{vmatrix}$$

$$= (1+2\cos\theta)\begin{vmatrix} 0 & \cos\theta-1 & 0 \\ 0 & 1-\cos\theta & \cos\theta-1 \\ 1 & \cos\theta & 1 \end{vmatrix} \qquad \text{(Operating } R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3\text{)}$$

$$= (1+2\cos\theta)(1-\cos\theta)^2 \qquad \text{(Expanding along } C_1\text{)}$$

$\therefore$ $$[\vec{a}\ \vec{b}\ \vec{c}] = (1-\cos\theta)\sqrt{1+2\cos\theta}$$

Thus, we get

$$p = \frac{1}{\sqrt{1+2\cos\theta}},\ q = \frac{-2\cos\theta}{\sqrt{1+2\cos\theta}},\ r = \frac{1}{\sqrt{1+2\cos\theta}}$$

EXAMPLE 3.19

Let V be the volume of the parallelepiped formed by the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If a_r, b_r and c_r, where $r = 1, 2, 3$, are non-negative real numbers and $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$, show that $V \le L^3$.

Sol. Given that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_2\hat{k}$$

where a_r, b_r, c_r $(r = 1, 2, 3)$ are all non-negative real numbers

Also $$\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$$

To prove $V \le L^3$, where V is the volume of the parallelepiped formed by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$$V = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\Rightarrow$ $= (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1)$ (i)

Now we know that A.M. $\ge$ G.M., therefore

$$\frac{(a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) + (a_3 + b_3 + c_3)}{3} \ge [(a_1 + b_1 + c_1)(a_2 + b_2 + c_2)(a_3 + b_3 + c_3)]^{1/3}$$

$\Rightarrow$ $[\vec{x} \times \vec{y}\ \vec{y} \times \vec{z}\ \vec{z} \times \vec{x}] \ge [(a_1 + b_1 + c_1)(a_2 + b_2 + c_2)(a_3 + b_3 + c_3)]^{1/3}$

$\Rightarrow$ $L^3 \ge (a_1 + b_1 + c_1)(a_2 + b_2 + c_2)(a_3 + b_3 + c_3)$

$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 +$ 24 more such terms

$\ge a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$ ($\because a_r, b_r, c_r \ge$ or $r = 1, 2, 3$)

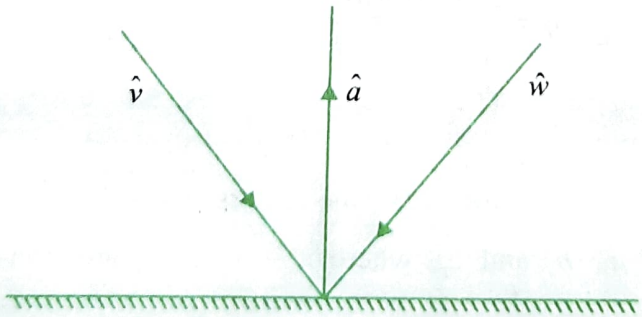
$\ge (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_2 + a_3 b_2 c_1)$ (same reason)

$= V$ [from (i)]

Thus, $L^3 \ge V$

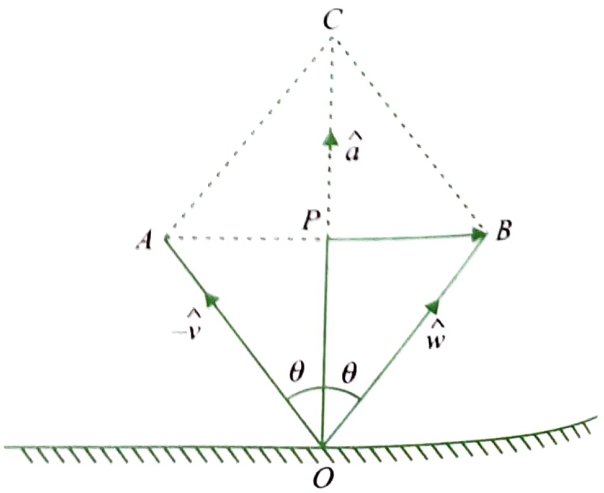
EXAMPLE 3.20

If the incident ray on a surface is along the unit vector $\hat{v}$, the reflected ray is along the unit vector $\hat{w}$ and the normal is along the unit vector $\hat{a}$ outwards, express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$.



Sol. Given that the incident ray is along $\hat{v}$, the reflected ray is along $\hat{w}$ and the normal is along $\hat{a}$, outwards. The given figure can be redrawn as shown in the figure.

We know that the incident ray, the reflected ray, and the normal lie in a plane, and the angle of incidence is equal to the angle of reflection.



Therefore, $\hat{a}$ will be along the angle bisector of $\hat{w}$ and $-\hat{v}$, i.e.,

$$\hat{a} = \frac{\hat{w} + (-\hat{v})}{|\hat{w} - \hat{v}|} \qquad \text{(i)}$$

But $\hat{a}$ is a unit vector

$$|\hat{w}-\hat{v}| = OC = 2OP = 2|\hat{w}|\cos\theta = 2\cos\theta$$

where

Substituting this value in (i), we get

$$\hat{a} = \frac{\hat{w}-\hat{v}}{2\cos\theta}$$

$$\hat{w} = \hat{v} + (2\cos\theta)\hat{a}$$

or

$$\hat{w} = \hat{v} - 2(\hat{a}\cdot\hat{v})\hat{a}$$

or

EXAMPLE 3.21

$\vec{u}, \vec{v}$ and $\vec{w}$ are three non-coplanar unit vectors and α, β and γ are the angles between $\vec{u}$ and $\vec{v}$, $\vec{v}$ and $\vec{w}$, and $\vec{w}$ and $\vec{u}$, respectively, and $\vec{x}$, $\vec{y}$ and $\vec{z}$ are unit vectors along the bisectors of the angles α, β and γ, respectively. Prove that $[\vec{x}\times\vec{y}\ \ \vec{y}\times\vec{z}\ \ \vec{z}\times\vec{x}] = \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$.

Sol. We know that $[\vec{x}\times\vec{y}\ \ \vec{y}\times\vec{z}\ \ \vec{z}\times\vec{x}] = [\vec{x}\ \vec{y}\ \vec{z}]^2$

Also a vector along the bisector of given two unit vectors $\vec{u}, \vec{v}$ is $\vec{u}+\vec{v}$.

A unit vector along the bisector is $\dfrac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|}$

$$|\vec{u}+\vec{v}|^2 = 1+1+2\vec{u}\cdot\vec{v} = 2+2\cos\alpha = 4\cos^2\frac{\alpha}{2}$$

$$\Rightarrow \quad \vec{x} = \frac{\vec{u}+\vec{v}}{2\cos\frac{\alpha}{2}}$$

Similarly, $\vec{y} = \dfrac{\vec{v}+\vec{w}}{2\cos\beta/2}$ and $\vec{z} = \dfrac{\vec{u}+\vec{w}}{2\cos\gamma/2}$

$$\Rightarrow \quad [\vec{x}\ \vec{y}\ \vec{z}] = \frac{1}{8}[\vec{u}+\vec{v}\ \ \vec{v}+\vec{w}\ \ \vec{u}+\vec{w}]\times\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= \frac{1}{8}2[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= \frac{1}{4}[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$\Rightarrow \quad [\vec{x}\times\vec{y}\ \ \vec{y}\times\vec{z}\ \ \vec{z}\times\vec{x}] = [\vec{x}\ \ \vec{y}\ \ \vec{z}]^2 = \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$

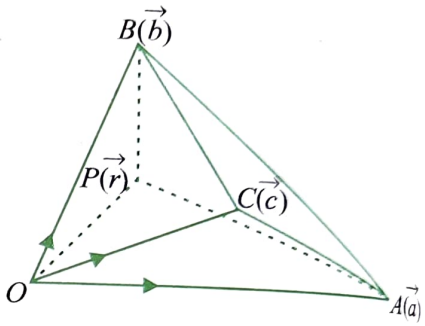
EXAMPLE 3.22

If $OABC$ is a tetrahedron where O is the origin and A, B and C are the other three vertices with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively, then prove that the centre of the sphere circumscribing the tetrahedron is given by position vector $\dfrac{a^2(\vec{b}\times\vec{c})+b^2(\vec{c}\times\vec{a})+c^2(\vec{a}\times\vec{b})}{2[\vec{a}\,\vec{b}\,\vec{c}]}$.

Sol. If the centre P is with position vector $\vec{r}$, then

$$\vec{a}-\vec{r}=\overrightarrow{PA},\ \vec{b}-\vec{r}=\overrightarrow{PB},\ \vec{c}-\vec{r}=\overrightarrow{PC},$$

where $|\overrightarrow{PA}|=|\overrightarrow{PB}|=|\overrightarrow{PC}|=|\overrightarrow{OP}|=|\vec{r}|$



Consider $|\vec{a}-\vec{r}|=|\vec{r}|$

or $(\vec{a}-\vec{r})\cdot(\vec{a}-\vec{r})=\vec{r}\cdot\vec{r}$

or $a^2-2\vec{a}\cdot\vec{r}+r^2=r^2$ or $a^2=2\vec{a}\cdot\vec{r}$

Similarly, $b^2=2\vec{b}\cdot\vec{r}$, $c^2=2\vec{c}\cdot\vec{r}$

Since $(\vec{b}\times\vec{c})$, $(\vec{c}\times\vec{a})$ and $(\vec{a}\times\vec{b})$ are non-coplanar, then

$$\vec{r}=x(\vec{b}\times\vec{c})+y(\vec{c}\times\vec{a})+z(\vec{a}\times\vec{b})$$

$$\Rightarrow\quad \vec{a}\cdot\vec{r}=x\vec{a}\cdot(\vec{b}\times\vec{c})+y\cdot0+z\cdot0$$

$$=x[\vec{a}\,\vec{b}\,\vec{c}]$$

or $$x=\frac{\vec{a}\cdot\vec{r}}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{a^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly, $y=\dfrac{b^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$ and $z=\dfrac{c^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$

Hence, $\vec{r}=\dfrac{a^2(\vec{b}\times\vec{c})+b^2(\vec{c}\times\vec{a})+c^2(\vec{a}\times\vec{b})}{2[\vec{a}\,\vec{b}\,\vec{c}]}$

EXAMPLE 3.23

Let k be the length of any edge of a regular tetrahedron (a tetrahedron whose edges are equal in length is called a regular tetrahedron). Show that the angle between any edge and a face not containing the edge is $\cos^{-1}(1/\sqrt{3})$.

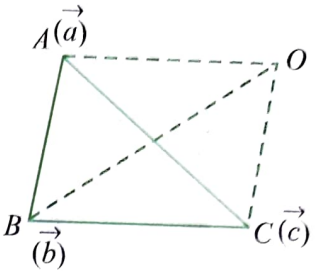
Sol. Let O be the origin of reference and A, B, C vertices with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively.

A vector normal to plane ABC is $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$ and $\overrightarrow{OA}=\vec{a}$.

The angle between a line and a plane is equal to the complement of the angle between the line and the normal to the plane. Thus, if θ denotes the angle between the face and edge, then

$$\sin\theta = \frac{(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})\cdot\vec{a}}{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|\cdot|\vec{a}|}$$

$$= \frac{\left|\begin{matrix}\vec{a} & \vec{b} & \vec{c}\end{matrix}\right|}{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|\cdot|\vec{a}|}$$



Now

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= k^6 \begin{vmatrix} 1 & \cos 60^\circ & \cos 60^\circ \\ \cos 60^\circ & 1 & \cos 60^\circ \\ \cos 60^\circ & \cos 60^\circ & 1 \end{vmatrix},$$

(where k is the length of the side of the tetrahedron)

$$= k^6\left(\frac{3}{4}-\frac{1}{8}-\frac{1}{8}\right)=\frac{1}{2}k^6$$

Also, $(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})$ is twice the area of triangle ABC, which is equilateral with each side k so that this is $\frac{\sqrt{3}}{2}k^2$. Hence,

$$\sin\theta = \frac{\frac{k^3}{\sqrt{2}}}{\frac{\sqrt{3}}{2}k^2\cdot k} = \frac{2}{\sqrt{6}} \Rightarrow \cos\theta = \frac{1}{\sqrt{3}}.$$

EXAMPLE 3.24

The lengths of two opposite edges of a tetrahedron are a and b; the shortest distance between these edges is d, and the angle between them is θ. Prove using vectors that the volume of the tetrahedron is $\frac{abd\sin\theta}{6}$.

Sol. In tetrahedron $OABC$, take O as the initial point and let the position vectors of A, B and C be $\vec{a}$, $\vec{k}$ and $\vec{c}$, respectively; then volume of the tetrahedron is equal to $\frac{1}{6}\vec{a}\cdot(\vec{k}\times\vec{c})$.

Also $\overrightarrow{BC} = \vec{c} - \vec{k}$ so that

$$V = \frac{1}{6}\vec{a} \cdot (\vec{k} \times (\vec{k} + \overrightarrow{BC}))$$

$$= \frac{1}{6}\vec{a} \cdot (\vec{k} \times \overrightarrow{BC})$$

$$= \frac{1}{6}\vec{k} \cdot (\overrightarrow{BC} \times \vec{a})$$

$$= \frac{1}{6}\vec{k} \cdot |BC|\,|a| \sin\theta \hat{n}$$

where $\hat{n}$ is the unit vector along PN, the line perpendicular to both OA and BC. Also $|BC| = b$.

Here $V = \frac{1}{6}ab\sin\theta\,\vec{k} \cdot \hat{n} = \frac{1}{6}ab\sin\theta$ (projection of OB on PN)

$\frac{1}{6}ab\sin\theta$ = (Perpendicular distance between OA and BC) $= \frac{1}{6}ab\sin\theta \cdot d = \frac{1}{6}abd\sin\theta$

Exercises

Single Correct Answer Type

1. Two vectors in space are equal only if they have equal component in
(1) a given direction
(2) two given directions
(3) three given directions
(4) in any arbitrary direction

2. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the three vectors having magnitudes 1, 5 and 3, respectively, such that the angle between $\vec{a}$ and $\vec{b}$ is θ and $\vec{a} \times (\vec{a} \times \vec{b}) = \vec{c}$. Then $\tan\theta$ is equal to
(1) 0 (2) 2/3 (3) 3/5 (4) 3/4

3. $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors of equal magnitude. The angle between each pair of vectors is $\pi/3$ such that $|\vec{a} + \vec{b} + \vec{c}| = \sqrt{6}$. Then $|\vec{a}|$ is equal to
(1) 2 (2) -1 (3) 1 (4) $\sqrt{6}/3$

4. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors, then the vector which is equally inclined to these vectors is
(1) $\vec{a} + \vec{b} + \vec{c}$
(2) $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}$
(3) $\frac{\vec{a}}{|\vec{a}|^2} + \frac{\vec{b}}{|\vec{b}|^2} + \frac{\vec{c}}{|\vec{c}|^2}$
(4) $|\vec{a}|\vec{a} - |\vec{b}|\vec{b} + |\vec{c}|\vec{c}$

5. Let $\vec{a} = \hat{i} + \hat{j}; \vec{b} = 2\hat{i} - \hat{k}$. Then vector $\vec{r}$ satisfying the equations $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ and $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is
(1) $\hat{i} - \hat{j} + \hat{k}$ (2) $3\hat{i} - \hat{j} + \hat{k}$ (3) $3\hat{i} + \hat{j} - \hat{k}$ (4) $\hat{i} - \hat{j} - \hat{k}$

6. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $\vec{a} \cdot \vec{b} < 0$ and $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$, then the angle between vectors $\vec{a}$ and $\vec{b}$ is
(1) π (2) $7\pi/4$ (3) $\pi/4$ (4) $3\pi/4$

7. If $\hat{a}, \hat{b}$ and $\hat{c}$ are three unit vectors, such that $\hat{a} + \hat{b} + \hat{c}$ is also a unit vector and θ_1, θ_2 and θ_3 are angles between the vectors $\hat{a}, \hat{b}$; $\hat{b}, \hat{c}$ and $\hat{c}, \hat{a}$, respectively, then among θ_1, θ_2 and θ_3
(1) all are acute angles
(2) all are right angles
(3) at least one is obtuse angle
(4) none of these

8. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} \cdot \vec{b} = 0 = \vec{a} \cdot \vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\pi/3$, then the value of $|\vec{a} \times \vec{b} - \vec{a} \times \vec{c}|$ is
(1) 1/2 (2) 1 (3) 2 (4) none of these

9. $P(\vec{p})$ and $Q(\vec{q})$ are the position vectors of two fixed points and $R(\vec{r})$ is the position vector of a variable point. If R moves such that $(\vec{r}-\vec{p})\times(\vec{r}-\vec{q})=\vec{0}$, then the locus of R is
(1) a plane containing the origin O and parallel to two non-collinear vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$
(2) the surface of a sphere described on PQ as its diameter
(3) a line passing through points P and Q
(4) a set of lines parallel to line PQ

10. Two adjacent sides of a parallelogram $ABCD$ are $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$. Then the value of $|\overrightarrow{AC}\times\overrightarrow{BD}|$ is
(1) $20\sqrt{5}$ (2) $22\sqrt{5}$ (3) $24\sqrt{5}$ (4) $26\sqrt{5}$

11. If $\hat{a},\hat{b}$ and $\hat{c}$ are three unit vectors inclined to each other at an angle θ, then the maximum value of θ is
(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{2}$ (3) $\frac{2\pi}{3}$ (4) $\frac{5\pi}{6}$

12. Let the pairs $\vec{a},\vec{b}$ and $\vec{c},\vec{d}$ each determine a plane. Then the planes are parallel if
(1) $(\vec{a}\times\vec{c})\times(\vec{b}\times\vec{d})=\vec{0}$ (2) $(\vec{a}\times\vec{c})\cdot(\vec{b}\times\vec{d})=\vec{0}$
(3) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{0}$ (4) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=\vec{0}$

13. If $\vec{r}\cdot\vec{a}=\vec{r}\cdot\vec{b}=\vec{r}\cdot\vec{c}=0$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar, then
(1) $\vec{r}\perp(\vec{c}\times\vec{a})$ (2) $\vec{r}\perp(\vec{a}\times\vec{b})$ (3) $\vec{r}\perp(\vec{b}\times\vec{c})$ (4) $\vec{r}=\vec{0}$

14. If $\vec{a}$ satisfies $\vec{a}\times(\hat{i}+2\hat{j}+\hat{k})=\hat{i}-\hat{k}$, then $\vec{a}$ is equal to
(1) $\lambda\hat{i}+(2\lambda-1)\hat{j}+\lambda\hat{k},\lambda\in R$ (2) $\lambda\hat{i}+(1-2\lambda)\hat{j}+\lambda\hat{k},\lambda\in R$
(3) $\lambda\hat{i}+(2\lambda+1)\hat{j}+\lambda\hat{k},\lambda\in R$ (4) $\lambda\hat{i}-(1+2\lambda)\hat{j}+\lambda\hat{k},\lambda\in R$

15. Vectors $3\vec{a}-5\vec{b}$ and $2\vec{a}+\vec{b}$ are mutually perpendicular. If $\vec{a}+4\vec{b}$ and $\vec{b}-\vec{a}$ are also mutually perpendicular, then the cosine of the angle between $\vec{a}$ and $\vec{b}$ is
(1) $\frac{19}{5\sqrt{43}}$ (2) $\frac{19}{3\sqrt{43}}$ (3) $\frac{19}{2\sqrt{45}}$ (4) $\frac{19}{6\sqrt{43}}$

16. The unit vector orthogonal to vector $-\hat{i}+2\hat{j}+2\hat{k}$ and making equal angles with the x- and y-axes is
(1) $\pm\frac{1}{3}(2\hat{i}+2\hat{j}-\hat{k})$ (2) $\pm\frac{1}{3}(\hat{i}+\hat{j}-\hat{k})$ (3) $\pm\frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$ (4) None of these

17. The value of x for which the angle between $\vec{a}=2x^2\hat{i}+4x\hat{j}+\hat{k}$ and $\vec{b}=7\hat{i}-2\hat{j}+x\hat{k}$ is obtuse and the angle between $\vec{b}$ and the z-axis is acute and less than $\pi/6$ is
(1) $a<x<1/2$ (2) $1/2<x<15$ (3) $x>1/2$ or $x<0$ (4) none of these

18. If vectors $\vec{a}$ and $\vec{b}$ are two adjacent sides of a parallelogram, then the vector representing the altitude of the parallelogram which is perpendicular to $\vec{a}$ is

(1) $\vec{b}+\frac{\vec{b}\times\vec{a}}{|\vec{a}|^2}$ (2) $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$ (3) $\vec{b}-\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$ (4) $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{|\vec{b}|^2}$

19. A parallelogram is constructed on $3\vec{a}+\vec{b}$ and $\vec{a}-4\vec{b}$, where $|\vec{a}|=6$ and $|\vec{b}|=8$, and $\vec{a}$ and $\vec{b}$ are anti-parallel. Then the length of the longer diagonal is

(1) 40 (2) 64 (3) 32 (4) 48

20. Let $\vec{a}\cdot\vec{b}=0$, where $\vec{a}$ and $\vec{b}$ are unit vectors and the unit vector $\vec{c}$ is inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c}=m\vec{a}+n\vec{b}+p(\vec{a}\times\vec{b})$, $(m, n, p \in R)$, then

(1) $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$ (2) $\frac{\pi}{4}\le\theta\le\frac{3\pi}{4}$ (3) $0\le\theta\le\frac{\pi}{4}$ (4) $0\le\theta\le\frac{3\pi}{4}$

21. $\vec{a}$ and $\vec{c}$ are unit vectors and $|\vec{b}|=4$. The angle between $\vec{a}$ and $\vec{c}$ is $\cos^{-1}(1/4)$ and $\vec{b}-2\vec{c}=\lambda\vec{a}$. The value of λ is

(1) 3, – 4 (2) 1/4, 3/4 (3) –3, 4 (4) –1/4, 3/4

22. Let the position vectors of the points P and Q be $4\hat{i}+\hat{j}+\lambda\hat{k}$ and $2\hat{i}-\hat{j}+\hat{k}$, respectively. Vector $\hat{i}-\hat{j}+6\hat{k}$ is perpendicular to the plane containing the origin and the points P and Q. Then λ equals

(1) – 1/2 (2) 1/2 (3) 1 (4) none of these

23. A vector of magnitude $\sqrt{2}$ coplanar with the vectors $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to the vector $\vec{c}=\hat{i}+\hat{j}+\hat{k}$, is

(1) $-\hat{j}+\hat{k}$ (2) $\hat{i}-\hat{k}$ (3) $\hat{i}-\hat{j}$ (4) $\hat{i}-\hat{j}$

24. Let P be a point interior to the acute triangle ABC. If $\vec{PA}+\vec{PB}+\vec{PC}$ is a null vector, then w.r.t. triangle ABC, point P is its

(1) centroid (2) orthocentre (3) incentre (4) circumcentre

25. G is the centroid of triangle ABC and A_1 and B_1 are the midpoints of sides AB and AC, respectively. If Δ_1 is the area of quadrilateral GA_1AB_1 and Δ is the area of triangle ABC, then Δ/Δ_1 is equal to

(1) $\frac{3}{2}$ (2) 3 (3) $\frac{1}{3}$ (4) none of these

26. Points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar and $(\sin\alpha)\vec{a}+(2\sin 2\beta)\vec{b}+(3\sin 3\gamma)\vec{c}-\vec{d}=\vec{0}$. Then the least value of $\sin^2\alpha+\sin^2 2\beta+\sin^2 3\gamma$ is

(1) 1/14 (2) 14 (3) 6 (4) $1/\sqrt{6}$

27. If $\vec{a}$ and $\vec{b}$ are any two vectors of magnitudes 1 and 2, respectively, and $(1-3\vec{a}\cdot\vec{b})^2+|2\vec{a}+\vec{b}+3(\vec{a}\times\vec{b})|^2=47$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\pi/3$ (2) $\pi-\cos^{-1}(1/4)$ (3) $\frac{2\pi}{3}$ (4) $\cos^{-1}(1/4)$

28. If $\vec{a}$ and $\vec{b}$ are any two vectors of magnitudes 2 and 3, respectively, such that $|2(\vec{a}\times\vec{b})|+|3(\vec{a}\cdot\vec{b})|=k$, then the maximum value of k is

(1) $\sqrt{13}$ (2) $2\sqrt{13}$ (3) $6\sqrt{13}$ (4) $10\sqrt{13}$

29. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $|\vec{a}+\vec{b}+3\vec{c}|=4$. Angle between $\vec{a}$ and $\vec{b}$ is θ_1, between $\vec{b}$ and $\vec{c}$ is θ_2 and between $\vec{a}$ and $\vec{c}$ varies $[\pi/6, 2\pi/3]$. Then the maximum value of $\cos\theta_1 + 3\cos\theta_2$ is

(1) 3 (2) 4 (3) $2\sqrt{2}$ (4) 6

30. If the vector product of a constant vector $\overrightarrow{OA}$ with a variable vector $\overrightarrow{OB}$ in a fixed plane OAB be a constant vector, then the locus of B is

(1) a straight line perpendicular to $\overrightarrow{OA}$ (2) a circle with centre O and radius equal to $|\overrightarrow{OA}|$

(3) a straight line parallel to $\overrightarrow{OA}$ (4) none of these

31. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be such that $|\vec{u}|=1, |\vec{v}|=2$ and $|\vec{w}|=3$. If the projection of $\vec{v}$ along $\vec{u}$ is equal to that of $\vec{w}$ along $\vec{u}$ and vectors $\vec{v}$ and $\vec{w}$ are perpendicular to each other, then $|\vec{u}-\vec{v}+\vec{w}|$ equals

(1) 2 (2) $\sqrt{7}$ (3) $\sqrt{14}$ (4) 14

32. If the two adjacent sides of two rectangles are represented by vectors $\vec{p}=5\vec{a}-3\vec{b}$; $\vec{q}=-\vec{a}-2\vec{b}$ and $\vec{r}=-4\vec{a}-\vec{b}$; $\vec{s}=-\vec{a}+\vec{b}$, respectively, then the angle between the vectors $\vec{x}=\frac{1}{3}(\vec{p}+\vec{r}+\vec{s})$ and $\vec{y}=\frac{1}{5}(\vec{r}+\vec{s})$ is

(1) $-\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ (2) $\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ (3) $\pi\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ (4) cannot be evaluated

33. If $\vec{\alpha}\|(\vec{\beta}\times\vec{\gamma})$, then $(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\gamma})$ equals to

(1) $|\vec{\alpha}|^2(\vec{\beta}\cdot\vec{\gamma})$ (2) $|\vec{\beta}|^2(\vec{\gamma}\cdot\vec{\alpha})$ (3) $|\vec{\gamma}|^2(\vec{\alpha}\cdot\vec{\beta})$ (4) $|\vec{\alpha}||\vec{\beta}||\vec{\gamma}|$

34. The position vectors of points A, B and C are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+5\hat{j}-\hat{k}$ and $2\hat{i}+3\hat{j}+5\hat{k}$, respectively. The greatest angle of triangle ABC is

(1) 120° (2) 90° (3) $\cos^{-1}(3/4)$ (4) none of these

35. Given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, two of which are non-collinear. Further if $(\vec{a}+\vec{b})$ is collinear with $\vec{c}$, $(\vec{b}+\vec{c})$ is collinear with $\vec{a}$, $|\vec{a}|=|\vec{b}|=|\vec{c}|=\sqrt{2}$. Find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

(1) 3 (2) −3 (3) 0 (4) cannot be evaluated

36. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $(\vec{a}+\vec{b})\cdot(2\vec{a}+3\vec{b})\times(3\vec{a}-2\vec{b})=\vec{0}$, then angle between $\vec{a}$ and $\vec{b}$ is

(1) 0 (2) $\pi/2$ (3) π (4) indeterminate

37. If in a right-angled triangle ABC, the hypotenuse $AB = p$, then $\overrightarrow{AB} \cdot \overrightarrow{AC} + \overrightarrow{BC} \cdot \overrightarrow{BA} + \overrightarrow{CA} \cdot \overrightarrow{CB}$ is equal to

(1) $2p^2$ (2) $\frac{p^2}{2}$ (3) p^2 (4) none of these

38. Resolved part of vector $\vec{a}$ and along vector $\vec{b}$ is $\vec{a}_1$ and that perpendicular to $\vec{b}$ is $\vec{a}_2$, then $\vec{a}_1 \times \vec{a}_2$ is equal to

(1) $\frac{(\vec{a} \times \vec{b}) \cdot \vec{b}}{|\vec{b}|^2}$ (2) $\frac{(\vec{a} \cdot \vec{b})\vec{a}}{|\vec{a}|^2}$ (3) $\frac{(\vec{a} \cdot \vec{b})(\vec{b} \times \vec{a})}{|\vec{b}|^2}$ (4) $\frac{(\vec{a} \cdot \vec{b})(\vec{b} \times \vec{a})}{|\vec{b} \times \vec{a}|}$

39. $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = \hat{i} + \hat{j} - 2\hat{k}$. A vector coplanar with $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is magnitude $\sqrt{\frac{2}{3}}$ is

(1) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (2) $-2\hat{i} - \hat{j} + 5\hat{k}$ (3) $2\hat{i} + 3\hat{j} + 3\hat{k}$ (4) $2\hat{i} + \hat{j} + 5\hat{k}$

40. If P is any arbitrary point on the circumcircle of the equilateral triangle of side length l units, then $|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2$ is always equal to

(1) $2l^2$ (2) $2\sqrt{3}\,l^2$ (3) l^2 (4) $3l^2$

41. If $\vec{r}$ and $\vec{s}$ are non-zero constant vectors and the scalar b is chosen such that $|\vec{r} + b\vec{s}|$ is minimum, then the value of $|b\vec{s}|^2 + |\vec{r} + b\vec{s}|^2$ is equal to

(1) $2|\vec{r}|^2$ (2) $|\vec{r}|^2/2$ (3) $3|\vec{r}|^2$ (4) $|\vec{r}|^2$

42. $\vec{a}$ and $\vec{b}$ are two unit vectors that are mutually perpendicular. A unit vector that is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$ is

(1) $\frac{1}{\sqrt{2}}(\vec{a} + \vec{b} + \vec{a} \times \vec{b})$ (2) $\frac{1}{2}(\vec{a} \times \vec{b} + \vec{a} + \vec{b})$

(3) $\frac{1}{\sqrt{3}}(\vec{a} + \vec{b} + \vec{a} \times \vec{b})$ (4) $\frac{1}{3}(\vec{a} + \vec{b} + \vec{a} \times \vec{b})$

43. Given that $\vec{a}, \vec{b}, \vec{p}, \vec{q}$ are four vectors such that $\vec{a} + \vec{b} = \mu\vec{p}, \vec{b} \cdot \vec{q} = 0$ and $(\vec{b})^2 = 1$, where μ is a scalar. Then $|(\vec{a} \cdot \vec{q})\vec{p} - (\vec{p} \cdot \vec{q})\vec{a}|$ is equal to

(1) $2|\vec{p} \cdot \vec{q}|$ (2) $(1/2)|\vec{p} \cdot \vec{q}|$ (3) $|\vec{p} \times \vec{q}|$ (4) $|\vec{p} \cdot \vec{q}|$

44. The position vectors of the vertices A, B and C of a triangle are three unit vectors $\hat{a}, \hat{b}$ and $\hat{c}$, respectively. A vector $\vec{d}$ is such that $\vec{d} \cdot \hat{a} = \vec{d} \cdot \hat{b} = \vec{d} \cdot \hat{c}$ and $\vec{d} = \lambda(\hat{b} + \hat{c})$. Then triangle ABC is

(1) acute angled (2) obtuse angled (3) right angled (4) none of these

45. If a is a real constant and A, B and C are variable angles and $\sqrt{a^2 - 4}\tan A + a\tan B + \sqrt{a^2 + 4}\tan C = 6a$, then the least value of $\tan^2 A + \tan^2 B + \tan^2 C$ is

(1) 6 (2) 10 (3) 12 (4) 3

46. The vertex A of triangle ABC is on the line $\vec{r} = \hat{i} + \hat{j} + \lambda\hat{k}$ and the vertices B and C have respective position vectors $\hat{i}$ and $\hat{j}$. Let Δ be the area of the triangle and $\Delta \in [3/2, \sqrt{33}/2]$. Then the range of values of λ corresponding to A is

(1) $[-8, -4] \cup [4, 8]$ (2) $[-4, 4]$ (3) $[-2, 2]$ (4) $[-4, -2] \cup [2, 4]$

47. A non-zero vector $\vec{a}$ is such that its projections along vectors $\frac{\hat{i}+\hat{j}}{\sqrt{2}}, \frac{-\hat{i}+\hat{j}}{\sqrt{2}}$ and $\hat{k}$ are equal, then unit vector along $\vec{a}$ is

(1) $\frac{\sqrt{2}\,\hat{j} - \hat{k}}{\sqrt{3}}$ (2) $\frac{\hat{j} - \sqrt{2}\hat{k}}{\sqrt{3}}$ (3) $\frac{\sqrt{2}}{\sqrt{3}}\hat{j} + \frac{\hat{k}}{\sqrt{3}}$ (4) $\frac{\hat{j} - \hat{k}}{\sqrt{2}}$

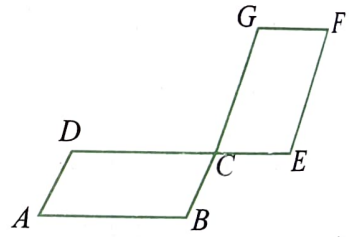
48. Position vector $\hat{k}$ is rotated about the origin by angle 135° in such a way that the plane made by it bisects the angle between $\hat{i}$ and $\hat{j}$. Then its new position is

(1) $\pm\frac{\hat{i}}{\sqrt{2}} \pm \frac{\hat{j}}{\sqrt{2}}$ (2) $\pm\frac{\hat{i}}{2} \pm \frac{\hat{j}}{2} - \frac{\hat{k}}{\sqrt{2}}$ (3) $\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{k}}{\sqrt{2}}$ (4) none of these

49. In a quadrilateral $ABCD$, $\overrightarrow{AC}$ is the bisector of $\overrightarrow{AB}$ and $\overrightarrow{AD}$, angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is $2\pi/3$, $15|\overrightarrow{AC}| = 3|\overrightarrow{AB}| = 5|\overrightarrow{AD}|$. Then the angle between $\overrightarrow{BA}$ and $\overrightarrow{CD}$ is

(1) $\cos^{-1}\frac{\sqrt{14}}{7\sqrt{2}}$ (2) $\cos^{-1}\frac{\sqrt{21}}{7\sqrt{3}}$ (3) $\cos^{-1}\frac{2}{\sqrt{7}}$ (4) $\cos^{-1}\frac{2\sqrt{7}}{14}$

50. In the figure, AB, DE and GF are parallel to each other and AD, BG and EF are parallel to each other. If $CD : CE = CG : CB = 2 : 1$, then the value of area (ΔAEG) : area (ΔABD) is equal to



(1) 7/2 (2) 3
(3) 4 (4) 9/2

51. Vector $\hat{a}$ in the plane of $\vec{b} = 2\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$ is such that it is equally inclined to $\vec{b}$ and $\vec{d}$ where $\vec{d} = \hat{j} + 2\hat{k}$. The value of $\hat{a}$ is

(1) $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ (2) $\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$ (3) $\frac{2\hat{i}+\hat{j}}{\sqrt{5}}$ (4) $\frac{2\hat{i}+\hat{j}}{\sqrt{5}}$

52. Let $ABCD$ be a tetrahedron such that the edges AB, AC and AD are mutually perpendicular. Let the area of triangles ABC, ACD and ADB be 3, 4 and 5 sq. units, respectively. Then the area of triangle BCD is

(1) $5\sqrt{2}$ (2) 5 (3) $\frac{\sqrt{5}}{2}$ (4) $\frac{5}{2}$

53. Let $\vec{f}(t) = [t]\hat{i} + (t - [t])\hat{j} + [t+1]\hat{k}$, where $[.]$ denotes the greatest integer function. Then the vectors $\vec{f}\left(\frac{5}{4}\right)$ and $\vec{f}(t)$, $0 < t < 1$, are

(1) parallel to each other (2) perpendicular to each other
(3) inclined at $\cos^{-1}\frac{2}{\sqrt{7(1-t^2)}}$ (4) inclined at $\cos^{-1}\frac{8+t}{9\sqrt{1+t^2}}$

54. If $\vec{a}$ is parallel to $\vec{b} \times \vec{c}$, then $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$ is equal to

(1) $|\vec{a}|^2 (\vec{b} \cdot \vec{c})$ (2) $|\vec{b}|^2 (\vec{a} \cdot \vec{c})$ (3) $|\vec{c}|^2 (\vec{a} \cdot \vec{b})$ (4) none of these

55. Three vectors $\hat{i} + \hat{j}$, $\hat{j} + \hat{k}$ and $\hat{k} + \hat{i}$ taken two at a time form three planes. The three unit vectors drawn perpendicular to these three planes form a parallelepiped of volume

(1) 1/3 (2) 4 (3) $(3\sqrt{3})/4$ (4) $4\sqrt{3}$

56. If $\vec{d} = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is a non-zero vector and $|(\vec{d} \cdot \vec{c})(\vec{a} \times \vec{b}) + (\vec{d} \cdot \vec{a})(\vec{b} \times \vec{c}) + (\vec{d} \cdot \vec{b})(\vec{c} \times \vec{a})| = 0$, then

(1) $|\vec{a}| = |\vec{b}| = |\vec{c}|$ (2) $|\vec{a}| + |\vec{b}| + |\vec{c}| = |\vec{d}|$

(3) $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar (4) none of these

57. If $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 0$, then $(\vec{a} \times (\vec{a} \times (\vec{a} \times (\vec{a} \times \vec{b}))))$ is equal to

(1) $48\hat{b}$ (2) $-48\hat{b}$ (3) $48\hat{a}$ (4) $-48\hat{a}$

58. If the two diagonals of one of its faces are $6\hat{i} + 6\hat{k}$ and $4\hat{j} + 2\hat{k}$ and of the edges not containing the given diagonals is $\vec{c} = 4\hat{j} - 8\hat{k}$, then the volume of a parallelepiped is

(1) 60 (2) 80 (3) 100 (4) 120

59. The volume of a tetrahedron formed by the coterminus edges $\vec{a}, \vec{b}$ and $\vec{c}$ is 3. Then the volume of the parallelepiped formed by the coterminus edges $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ is

(1) 6 (2) 18 (3) 36 (4) 9

60. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually orthogonal unit vectors, then the triple product $[\vec{a} + \vec{b} + \vec{c} \;\; \vec{a} + \vec{b} \;\; \vec{b} + \vec{c}]$ equals

(1) 0 (2) 1 or -1 (3) 1 (4) 3

61. Vector $\vec{c}$ is perpendicular to vectors $\vec{a} = (2, -3, 1)$ and $\vec{b} = (1, -2, 3)$ and satisfies the condition $\vec{c} \cdot (\hat{i} + 2\hat{j} - 7\hat{k}) = 10$. Then vector $\vec{c}$ is equal to

(1) (7, 5, 1) (2) $(-7, -5, -1)$ (3) $(1, 1, -1)$ (4) none of these

62. Given $\vec{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j}$; $\vec{a} \perp \vec{b}$, $\vec{a} \cdot \vec{c} = 4$. Then

(1) $[\vec{a} \; \vec{b} \; \vec{c}]^2 = |\vec{a}|$ (2) $[\vec{a} \; \vec{b} \; \vec{c}] = |\vec{a}|$ (3) $[\vec{a} \; \vec{b} \; \vec{c}] = 0$ (4) $[\vec{a} \; \vec{b} \; \vec{c}] = |\vec{a}|^2$

63. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, then the value of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$ is

(1) 0 (2) 1

(3) $\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$ (4) $\frac{3}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

64. Let $\vec{r}, \vec{a}, \vec{b}$ and $\vec{c}$ be four non-zero vectors such that $\vec{r}\cdot\vec{a}=0$, $|\vec{r}\times\vec{b}|=|\vec{r}||\vec{b}|$ and $|\vec{r}\times\vec{c}|=|\vec{r}||\vec{c}|$. Then $[a\ b\ c]$ is equal to

(1) $|a|\,|b|\,|c|$ (2) $-|a|\,|b|\,|c|$ (3) 0 (4) none of these

65. If $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $[\vec{a}\,\vec{b}\,\vec{c}]=1$, $\vec{c}=\lambda\vec{a}\times\vec{b}$, angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$, $|\vec{a}|=\sqrt{2}$, $|\vec{b}|=\sqrt{3}$ and $|\vec{c}|=\frac{1}{\sqrt{3}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

66. If $4\vec{a}+5\vec{b}+9\vec{c}=0$, then $(\vec{a}\times\vec{b})\times[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]$ is equal to

(1) a vector perpendicular to the plane of $\vec{a}, \vec{b}$ and $\vec{c}$ (2) a scalar quantity

(3) $\vec{0}$ (4) none of these

67. Value of $[\vec{a}\times\vec{b}\ \ \vec{a}\times\vec{c}\ \ \vec{d}]$ is always equal to

(1) $(\vec{a}\cdot\vec{d})[\vec{a}\,\vec{b}\,\vec{c}]$ (2) $(\vec{a}\cdot\vec{c})[\vec{a}\,\vec{b}\,\vec{d}]$ (3) $(\vec{a}\cdot\vec{b})[\vec{a}\,\vec{b}\,\vec{d}]$ (4) none of these

68. Let $\hat{a}$ and $\hat{b}$ be mutually perpendicular unit vectors. Then for any arbitrary $\vec{r}$,

(1) $\vec{r}=(\vec{r}\cdot\hat{a})\hat{a}+(\vec{r}\cdot\hat{b})\hat{b}+(\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

(2) $\vec{r}=(\vec{r}\cdot\hat{a})-(\vec{r}\cdot\hat{b})\hat{b}-(\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

(3) $\vec{r}=(\vec{r}\cdot\hat{a})\hat{a}-(\vec{r}\cdot\hat{b})\hat{b}+(\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

(4) none of these

69. Let $\vec{a}$ and $\vec{b}$ be unit vectors perpendicular to each other. Then $[\vec{a}+(\vec{a}\times\vec{b})\ \ \vec{b}+(\vec{a}\times\vec{b})\ \ \vec{a}\times\vec{b}]$ will always be equal to

(1) 1 (2) 0 (3) -1 (4) none of these

70. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}|=1, |\vec{b}|=4$ and $\vec{a}\cdot\vec{b}=2$. If $\vec{c}=(2\vec{a}\times\vec{b})-3\vec{b}$, then find the angle between $\vec{b}$ and $\vec{c}$.

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{3\pi}{4}$ (4) $\frac{5\pi}{6}$

71. $\vec{b}$ and $\vec{c}$ are unit vectors. Then for any arbitrary vector $\vec{a}$, $(((\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c}))\cdot(\vec{b}-\vec{c})$ is always equal to

(1) $|\vec{a}|$ (2) $\frac{1}{2}|\vec{a}|$ (3) $\frac{1}{3}|\vec{a}|$ (4) none of these

72. If $\vec{a}.\vec{b} = \beta$ and $\vec{a} \times \vec{b} = \vec{c}$, then $\vec{b}$ is

(1) $\dfrac{(\beta\vec{a} - \vec{a} \times \vec{c})}{|\vec{a}|^2}$ (2) $\dfrac{(\beta\vec{a} + \vec{a} \times \vec{c})}{|\vec{a}|^2}$ (3) $\dfrac{(\beta\vec{c} - \vec{a} \times \vec{c})}{|\vec{a}|^2}$ (4) $\dfrac{(\beta\vec{a} + \vec{a} \times \vec{c})}{|\vec{a}|^2}$

73. If $a(\vec{\alpha} \times \vec{\beta}) + b(\vec{\beta} \times \vec{\gamma}) + c(\vec{\gamma} \times \vec{\alpha}) = 0$ and at least one of a, b and c is non-zero, then vectors $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are

(1) parallel (2) coplanar (3) mutually perpendicular (4) none of these

74. If $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}) = \vec{b}$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors, then

(1) $\vec{a}, \vec{b}$ and $\vec{c}$ can be coplanar (2) $\vec{a}, \vec{b}$ and $\vec{c}$ must be coplanar

(3) $\vec{a}, \vec{b}$ and $\vec{c}$ cannot be coplanar (4) none of these

75. If $\vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot\vec{c} = \dfrac{1}{2}$ for some non-zero vector $\vec{r}$, then the area of the triangle whose vertices are $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ is ($\vec{a}, \vec{b}, \vec{c}$ are non-coplanar)

(1) $|[\vec{a}\,\vec{b}\,\vec{c}]|$ (2) $|\vec{r}|$ (3) $|[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}|$ (4) none of these

76. A vector of magnitude 10 along the normal to the curve $3x^2 + 8xy + 2y^2 - 3 = 0$ at its point $P(1, 0)$ can be

(1) $6\hat{i} + 8\hat{j}$ (2) $-8\hat{i} + 3\hat{j}$ (3) $6\hat{i} - 8\hat{j}$ (4) $8\hat{i} + 6\hat{j}$

77. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle $\pi/3$, then $\{\vec{a} \times (\vec{b} + \vec{a} \times \vec{b})\} \cdot \vec{b}$ is equal to

(1) $\dfrac{-3}{4}$ (2) $\dfrac{1}{4}$ (3) $\dfrac{3}{4}$ (4) $\dfrac{1}{2}$

78. If $\vec{a}$ and $\vec{b}$ are orthogonal unit vectors, then for a vector $\vec{r}$ non-coplanar with $\vec{a}$ and $\vec{b}$, vector $\vec{r} \times \vec{a}$ is equal to

(1) $[\vec{r}\,\vec{a}\,\vec{b}]\vec{b} - (\vec{r}\cdot\vec{b})(\vec{b} \times \vec{a})$ (2) $[\vec{r}\,\vec{a}\,\vec{b}](\vec{a} + \vec{b})$

(3) $[\vec{r}\,\vec{a}\,\vec{b}]\vec{a} + (\vec{r}\cdot\vec{a})\vec{a} \times \vec{b}$ (4) none of these

79. If $\vec{a}, \vec{b}, \vec{c}$ are any three non-coplanar vectors, then the equation $[\vec{b} \times \vec{c}\ \vec{c} \times \vec{a}\ \vec{a} \times \vec{b}]\ x^2 + [\vec{a} + \vec{b}\ \vec{b} + \vec{c}\ \vec{c} + \vec{a}]x + 1 + [\vec{b} - \vec{c}\ \vec{c} - \vec{a}\ \vec{a} - \vec{b}] = 0$ has roots

(1) real and distinct (2) real (3) equal (4) imaginary

80. If $\vec{x} + \vec{c} \times \vec{y} = \vec{a}$ and $\vec{y} + \vec{c} \times \vec{x} = \vec{b}$, where $\vec{c}$ is a non-zero vector, then which of the following is not correct?

(1) $\vec{x} = \dfrac{\vec{b} \times \vec{c} + \vec{a} + (\vec{c}\cdot\vec{a})\vec{c}}{1 + \vec{c}\cdot\vec{c}}$ (2) $\vec{x} = \dfrac{\vec{c} \times \vec{b} + \vec{b} + (\vec{c}\cdot\vec{a})\vec{c}}{1 + \vec{c}\cdot\vec{c}}$

(3) $\vec{y} = \dfrac{\vec{a} \times \vec{c} + \vec{b} + (\vec{c}\cdot\vec{b})\vec{c}}{1 + \vec{c}\cdot\vec{c}}$ (4) none of these

81. The condition for equations $\vec{r} \times \vec{a} = \vec{b}$ and $\vec{r} \times \vec{c} = \vec{d}$ to be consistent is

(1) $\vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{d}$ (2) $\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{d}$ (3) $\vec{b} \cdot \vec{c} + \vec{a} \cdot \vec{d} = 0$ (4) $\vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{d} = 0$

82. If $\vec{a}$ and $\vec{b}$ are non-zero non-collinear vectors, then $[\vec{a}\ \vec{b}\ \hat{i}]\hat{i} + [\vec{a}\ \vec{b}\ \hat{j}]\hat{j} + [\vec{a}\ \vec{b}\ \hat{k}]\hat{k}$ is equal to

(1) $\vec{a} + \vec{b}$ (2) $\vec{a} \times \vec{b}$ (3) $\vec{a} - \vec{b}$ (4) $\vec{b} \times \vec{a}$

83. If $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{c} = \hat{i} + \hat{j} + 2\hat{k}$ and $(1+\alpha)\hat{i} + \beta(1+\alpha)\hat{j} + \gamma(1+\alpha)(1+\beta)\hat{k} = \vec{a} \times (\vec{b} \times \vec{c})$, then α, β and γ are

(1) $-2, -4, -\frac{2}{3}$ (2) $2, -4, \frac{2}{3}$ (3) $-2, 4, \frac{2}{3}$ (4) $2, 4, -\frac{2}{3}$

84. Let $\vec{a}(x) = (\sin x)\hat{i} + (\cos x)\hat{j}$ and $\vec{b}(x) = (\cos 2x)\hat{i} + (\sin 2x)\hat{j}$ be two variable vectors $(x \in R)$. Then $\vec{a}(x)$ and $\vec{b}(x)$ are

(1) collinear for unique value of x
(2) perpendicular for infinite values of x
(3) zero vectors for unique value of x
(4) none of these

85. For any two vectors $\vec{a}$ and $\vec{b}$, $(\vec{a} \times \hat{i}) \cdot (\vec{b} \times \hat{i}) + (\vec{a} \times \hat{j}) \cdot (\vec{b} \times \hat{j}) + (\vec{a} \times \hat{k}) \cdot (\vec{b} \times \hat{k})$ is always equal to

(1) $\vec{a} \cdot \vec{b}$ (2) $2\vec{a} \cdot \vec{b}$ (3) zero (4) none of these

86. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{r}$ be any arbitrary vector. Then $(\vec{a} \times \vec{b}) \times (\vec{r} \times \vec{c}) + (\vec{b} \times \vec{c}) \times (\vec{r} \times \vec{a}) + (\vec{c} \times \vec{a}) \times (\vec{r} \times \vec{b})$ is always equal to

(1) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (2) $2[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (3) $3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (4) none of these

87. If $\vec{p} = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{q} = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{r} = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then the value of the expression $(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{p} + \vec{q} + \vec{r})$ is

(1) 3 (2) 2 (3) 1 (4) 0

88. $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ are the vertices of triangle ABC and $R(\vec{r})$ is any point in the plane of triangle ABC, then $\vec{r} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$ is always equal to

(1) zero (2) $[\vec{a}\ \vec{b}\ \vec{c}]$ (3) $-[\vec{a}\ \vec{b}\ \vec{c}]$ (4) none of these

89. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors and $\vec{a} \times \vec{c}$ is perpendicular to $\vec{a} \times (\vec{b} \times \vec{c})$, then the value of $[\vec{a} \times (\vec{b} \times \vec{c})] \times \vec{c}$ is equal to

(1) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{c}$ (2) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{b}$ (3) $\vec{0}$ (4) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{a}$

90. If V be the volume of a tetrahedron and V' be the volume of another tetrahedran formed by the centroids of faces of the previous tetrahedron and $V = KV'$, then K is equal to

(1) 9 (2) 12 (3) 27 (4) 81

91. $[(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\ \ (\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\ \ (\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$ is equal to (where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-zero non-coplanar vectors)

(1) $[\vec{a}\,\vec{b}\,\vec{c}]^2$ (2) $[\vec{a}\,\vec{b}\,\vec{c}]^3$ (3) $[\vec{a}\,\vec{b}\,\vec{c}]^4$ (4) $[\vec{a}\,\vec{b}\,\vec{c}]$

92. If $\vec{r} = x_1(\vec{a}\times\vec{b}) + x_2(\vec{b}\times\vec{a}) + x_3(\vec{c}\times\vec{d})$ and $4[\vec{a}\,\vec{b}\,\vec{c}] = 1$, then $x_1 + x_2 + x_3$ is equal to

(1) $\frac{1}{2}\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ (2) $-\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ (3) $2\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ (4) $4\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$

93. If $\vec{a}\perp\vec{b}$, then vector $\vec{v}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying the equations $\vec{v}\cdot\vec{a} = 0$ and $\vec{v}\cdot\vec{b} = 1$ and $[\vec{v}\,\vec{a}\,\vec{b}] = 1$ is

(1) $\frac{\vec{b}}{|\vec{b}|^2} + \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$ (2) $\frac{\vec{b}}{|\vec{b}|} + \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$ (3) $\frac{\vec{b}}{|\vec{b}|^2} + \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ (4) none of these

94. If $\vec{a'} = \hat{i}+\hat{j}$, $\vec{b'} = \hat{i}-\hat{j}+2\hat{k}$ and $\vec{c'} = 2\hat{i}+\hat{j}-\hat{k}$, then the altitude of the parallelepiped formed by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ having base formed by $\vec{b}$ and $\vec{c}$ is (where $\vec{a'}$ is reciprocal vector $\vec{a}$, etc.)

(1) 1 (2) $3\sqrt{2}/2$ (3) $1/\sqrt{6}$ (4) $1/\sqrt{2}$

95. If $\vec{a} = \hat{i}+\hat{j}$, $\vec{b} = \hat{j}+\hat{k}$, $\vec{c} = \hat{k}+\hat{i}$, then in the reciprocal system of vectors $\vec{a}, \vec{b}, \vec{c}$ reciprocal $\vec{a}$ of vector $\vec{a}$ is

(1) $\frac{\hat{i}+\hat{j}+\hat{k}}{2}$ (2) $\frac{\hat{i}-\hat{j}+\hat{k}}{2}$ (3) $\frac{-\hat{i}-\hat{j}+\hat{k}}{2}$ (4) $\frac{\hat{i}+\hat{j}-\hat{k}}{2}$

Multiple Correct Answers Type

1. If unit vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle 2θ such that $|\vec{a}-\vec{b}| < 1$ and $0 \le \theta \le \pi$, then θ lies in the interval

(1) $[0, \pi/6)$ (2) $(5\pi/6, \pi]$ (3) $[\pi/6, \pi/2)$ (4) $(\pi/2, 5\pi/6]$

2. $\vec{b}$ and $\vec{c}$ are non-collinear if $\vec{a}\times(\vec{b}\times\vec{c}) + (\vec{a}\cdot\vec{b})\vec{b} = (4-2x-\sin y)\vec{b} + (x^2-1)\vec{c}$ and $(\vec{c}\cdot\vec{c})\vec{a} = \vec{c}$. Then

(1) $x = 1$ (2) $x = -1$ (3) $y = (4n+1)\frac{\pi}{2},\ n \in I$ (4) $y = (2n+1)\frac{\pi}{2},\ n \in I$

3. Unit vectors $\vec{a}$ and $\vec{b}$ are perpendicular, and unit vector $\vec{c}$ is inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})$, then

(1) $a = \beta$ (2) $\gamma^2 = 1 - 2\alpha^2$ (3) $\gamma^2 = -\cos 2\theta$ (4) $\beta^2 = \frac{1+\cos 2\theta}{2}$

4. $\vec{a}$ and $\vec{b}$ are two given vectors. With these vectors as adjacent sides, a parallelogram is constructed. The vector which is the altitude of the parallelogram and which is perpendicular to $\vec{a}$ is

(1) $\dfrac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\vec{a}-\vec{b}$ (2) $\dfrac{1}{|\vec{a}|^2}\{|\vec{a}|^2\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}\}$

(3) $\dfrac{\vec{a}\times(\vec{a}\times\vec{b})}{|\vec{a}|^2}$ (4) $\dfrac{\vec{a}\times(\vec{b}\times\vec{a})}{|\vec{b}|^2}$

5. If $\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to $(\vec{a}\times\vec{b})\times\vec{c}$, we may have

(1) $(\vec{a}\cdot\vec{c})|\vec{b}|^2=(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$ (2) $\vec{a}\cdot\vec{b}=0$

(3) $\vec{a}\cdot\vec{c}=0$ (4) $\vec{b}\cdot\vec{c}=0$

6. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors forming right-hand triad. Let $\vec{p}=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{q}=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{r}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

If $x\in R^+$, then

(1) $x[\vec{a}\,\vec{b}\,\vec{c}]+\dfrac{[\vec{p}\,\vec{q}\,\vec{r}]}{x}$ has least value 2 (2) $x^4[\vec{a}\,\vec{b}\,\vec{c}]^2+\dfrac{[\vec{p}\,\vec{q}\,\vec{r}]}{x^2}$ has least value $(3/2^{2/3})$

(3) $[\vec{p}\,\vec{q}\,\vec{r}]>0$ (4) none of these

7. $a_1, a_2, a_3\in R-\{0\}$ and $a_1+a_2\cos 2x+a_3\sin^2 x=0$ for all $x\in R$, then

(1) vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=4\hat{i}+2\hat{j}+\hat{k}$ are perpendicular to each other

(2) vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}+2\hat{k}$ are parallel to each other

(3) if vector $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ is of length $\sqrt{6}$ units, then one of the ordered tripplet $(a_1, a_2, a_3)=(1,-1,-2)$

(4) if $2a_1+3a_2+6a_3=26$, then $|a_1\hat{i}+a_2\hat{j}+a_3\hat{k}|$ is $2\sqrt{6}$

8. If $\vec{a}$ and $\vec{b}$ are two vectors and angle between them is θ, then

(1) $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$

(2) $|\vec{a}\times\vec{b}|=(\vec{a}\cdot\vec{b})$, if $\theta=\pi/4$

(3) $\vec{a}\times\vec{b}=(\vec{a}\cdot\vec{b})\hat{n}$ (where $\hat{n}$ is a normal unit vector), if $\theta=\pi/4$

(4) $(\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b})=0$

9. Let $\vec{a}$ and $\vec{b}$ be two non-zero perpendicular vectors. A vector $\vec{r}$ satisfying the equation $\vec{r}\times\vec{b}=\vec{a}$ can be

(1) $\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$ (2) $2\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$ (3) $|\vec{a}|\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$ (4) $|\vec{b}|\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$

10. If vectors $\vec{b} = (\tan\alpha, -1, 2\sqrt{\sin\alpha/2})$ and $\vec{c} = \left(\tan\alpha, \tan\alpha, -\frac{3}{\sqrt{\sin\alpha/2}}\right)$ are orthogonal and vector $\vec{a} = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the z-axis, then the value of α is

(1) $\alpha = (4n+1)\pi + \tan^{-1} 2$ (2) $\alpha = (4n+1)\pi - \tan^{-1} 2$

(3) $\alpha = (4n+2)\pi + \tan^{-1} 2$ (4) $\alpha = (4n+2)\pi - \tan^{-1} 2$

11. Let $\vec{r}$ be a unit vector satisfying $\vec{r} \times \vec{a} = \vec{b}$, where $|\vec{a}| = \sqrt{3}$ and $|\vec{b}| = \sqrt{2}$. Then

(1) $\vec{r} = \frac{2}{3}(\vec{a} + \vec{a} \times \vec{b})$ (2) $\vec{r} = \frac{1}{3}(\vec{a} + \vec{a} \times \vec{b})$

(3) $\vec{r} = \frac{2}{3}(\vec{a} - \vec{a} \times \vec{b})$ (4) $\vec{r} = \frac{1}{3}(-\vec{a} + \vec{a} \times \vec{b})$

12. If $\vec{a}$ and $\vec{b}$ are unequal unit vectors such that $(\vec{a} - \vec{b}) \times [(\vec{b} + \vec{a}) \times (2\vec{a} + \vec{b})] = \vec{a} + \vec{b}$, then angle θ between $\vec{a}$ and $\vec{b}$ is

(1) 0 (2) $\pi/2$ (3) $\pi/4$ (4) π

13. If $\vec{a}$ and $\vec{b}$ are two unit vectors perpendicular to each other and $\vec{c} = \lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3(\vec{a} \times \vec{b})$, then which of the following is (are) true?

(1) $\lambda_1 = \vec{a} \cdot \vec{c}$ (2) $\lambda_2 = |\vec{b} \times \vec{c}|$

(3) $\lambda_3 = |(\vec{a} \times \vec{b}) \times \vec{c}|$ (4) $\lambda_1 + \lambda_2 + \lambda_3 = (\vec{a} + \vec{b} + \vec{a} \times \vec{b}) \cdot \vec{c}$

14. If vectors $\vec{a}$ and $\vec{b}$ are non-collinear, then $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$ is

(1) a unit vector (2) in the plane of $\vec{a}$ and $\vec{b}$

(3) equally inclined to $\vec{a}$ and $\vec{b}$ (4) perpendicular to $\vec{a} \times \vec{b}$

15. If $\vec{a}$ and $\vec{b}$ are non-zero vectors such that $|\vec{a} + \vec{b}| = |\vec{a} - 2\vec{b}|$, then

(1) $2\vec{a} \cdot \vec{b} = |\vec{b}|^2$ (2) $\vec{a} \cdot \vec{b} = |\vec{b}|^2$

(3) least value of $\vec{a} \cdot \vec{b} + \frac{1}{|\vec{b}|^2 + 2}$ is $\sqrt{2}$ (4) least value of $\vec{a} \cdot \vec{b} + \frac{1}{|\vec{b}| + 2}$ is $\sqrt{2} - 1$

16. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-zero vectors and $\vec{V_1} = \vec{a} \times (\vec{b} \times \vec{c})$ and $\vec{V_2} = (\vec{a} \times \vec{b}) \times \vec{c}$. Vectors $\vec{V_1}$ and $\vec{V_2}$ are equal. Then

(1) $\vec{a}$ and $\vec{b}$ are orthogonal (2) $\vec{a}$ and $\vec{c}$ are collinear

(3) $\vec{b}$ and $\vec{c}$ are orthogonal (4) $\vec{b} = \lambda(\vec{a} \times \vec{c})$ when λ is a scalar

17. Vectors $\vec{A}$ and $\vec{B}$ satisfying the vector equation $\vec{A}+\vec{B}=\vec{a}$, $\vec{A}\times\vec{B}=\vec{b}$ and $\vec{A}\cdot\vec{a}=1$, where $\vec{a}$ and $\vec{b}$ are given vectors, are

(1) $\vec{A}=\dfrac{(\vec{a}\times\vec{b})-\vec{a}}{a^2}$

(2) $\vec{B}=\dfrac{(\vec{b}\times\vec{a})+\vec{a}(a^2-1)}{a^2}$

(3) $\vec{A}=\dfrac{(\vec{a}\times\vec{b})+\vec{a}}{a^2}$

(4) $\vec{B}=\dfrac{(\vec{b}\times\vec{a})-\vec{a}(a^2-1)}{a^2}$

18. A vector $\vec{d}$ is equally inclined to three vectors $\vec{a}=\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{j}-2\hat{k}$. Let $\vec{x}$, $\vec{y}$ and $\vec{z}$ be three vectors in the plane of $\vec{a}, \vec{b}$; $\vec{b}, \vec{c}$; $\vec{c}, \vec{a}$, respectively. Then

(1) $\vec{x}\cdot\vec{d}=-1$

(2) $\vec{y}\cdot\vec{d}=1$

(3) $\vec{z}\cdot\vec{d}=0$

(4) $\vec{r}\cdot\vec{d}=0$, where $\vec{r}=\lambda\vec{x}+\mu\vec{y}+\delta\vec{z}$

19. Vectors perpendicular to $\hat{i}-\hat{j}-\hat{k}$ and in the plane of $\hat{i}+\hat{j}+\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ are

(1) $\hat{i}+\hat{k}$ (2) $2\hat{i}+\hat{j}+\hat{k}$ (3) $3\hat{i}+2\hat{j}+\hat{k}$ (4) $-4\hat{i}-2\hat{j}-2\hat{k}$

20. If side $\overrightarrow{AB}$ of an equilateral triangle ABC lying in the x-y plane is $3\hat{i}$, then side $\overrightarrow{CB}$ can be

(1) $-\dfrac{3}{2}(\hat{i}-\sqrt{3}\hat{j})$ (2) $-\dfrac{3}{2}(\hat{i}-\sqrt{3}\hat{j})$ (3) $-\dfrac{3}{2}(\hat{i}+\sqrt{3}\hat{j})$ (4) $\dfrac{3}{2}(\hat{i}+\sqrt{3}\hat{j})$

21. The angles of a triangle, two of whose sides are represented by vectors $\sqrt{3}(\vec{a}\times\vec{b})$ and $\vec{b}-(\hat{a}\cdot\vec{b})\hat{a}$, where $\vec{b}$ is a non-zero vector and $\hat{a}$ is a unit vector in the direction of $\vec{a}$, are

(1) $\tan^{-1}(\sqrt{3})$ (2) $\tan^{-1}(1/\sqrt{3})$ (3) $\cot^{-1}(0)$ (4) $\tan^{-1}(1)$

22. $\vec{a}, \vec{b}$ and $\vec{c}$ are unimodular and coplanar. A unit vector $\vec{d}$ is perpendicular to them. If $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\dfrac{1}{6}\hat{i}-\dfrac{1}{3}\hat{j}+\dfrac{1}{3}\hat{k}$, and the angle between $\vec{a}$ and $\vec{b}$ is 30°, then $\vec{c}$ is

(1) $(\hat{i}-2\hat{j}+2\hat{k})/3$ (2) $(-\hat{i}+2\hat{j}-2\hat{k})/3$ (3) $(2\hat{i}+2\hat{j}-\hat{k})/3$ (4) $(-2\hat{i}-2\hat{j}+\hat{k})/3$

23. If $\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$, then $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=$

(1) $2(\vec{a}\times\vec{b})$ (2) $6(\vec{b}\times\vec{c})$ (3) $3(\vec{c}\times\vec{a})$ (4) $\vec{0}$

24. $\vec{a}$ and $\vec{b}$ are two non-collinear unit vectors, and $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$. Then $|\vec{v}|$ is

(1) $|\vec{u}|$ (2) $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ (3) $|\vec{u}|+|\vec{u}\cdot\vec{a}|$ (4) none of these

25. If $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{a}$, where $\vec{c}\neq\vec{0}$, then

(1) $|\vec{a}|=|\vec{c}|$ (2) $|\vec{a}|=|\vec{b}|$ (3) $|\vec{b}|=1$ (4) $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$

26. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{d}$ be a non-zero vector, which is perpendicular to $(\vec{a}+\vec{b}+\vec{c})$. Now $\vec{d} = (\vec{a}\times\vec{b})\sin x + (\vec{b}\times\vec{c})\cos y + 2(\vec{c}\times\vec{a})$. Then

(1) $\dfrac{\vec{d}\cdot(\vec{a}+\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]} = 2$

(2) $\dfrac{\vec{d}\cdot(\vec{a}+\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]} = -2$

(3) minimum value of x^2+y^2 is $\pi^2/4$

(4) minimum value of x^2+y^2 is $5\pi^2/4$

27. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \dfrac{1}{2}\vec{b}$, then ($\vec{b}$ and $\vec{c}$ being non-parallel)

(1) angle between $\vec{a}$ and $\vec{b}$ is $\pi/3$

(2) angle between $\vec{a}$ and $\vec{c}$ is $\pi/3$

(3) angle between $\vec{a}$ and $\vec{b}$ is $\pi/2$

(4) angle between $\vec{a}$ and $\vec{c}$ is $\pi/2$

28. If in triangle ABC, $\overrightarrow{AB} = \dfrac{\vec{u}}{|\vec{u}|} - \dfrac{\vec{v}}{|\vec{v}|}$ and $\overrightarrow{AC} = \dfrac{2\vec{u}}{|\vec{u}|}$, where $|\vec{u}| \neq |\vec{v}|$, then

(1) $1+\cos 2A+\cos 2B+\cos 2C = 0$

(2) $\sin A = \cos C$

(3) projection of AC on BC is equal to BC

(4) projection of AB on BC is equal to AB

29. $[\vec{a}\times\vec{b}\;\;\vec{c}\times\vec{d}\;\;\vec{e}\times\vec{f}]$ is equal to

(1) $[\vec{a}\,\vec{b}\,\vec{d}][\vec{c}\,\vec{e}\,\vec{f}] - [\vec{a}\,\vec{b}\,\vec{c}][\vec{d}\,\vec{e}\,\vec{f}]$

(2) $[\vec{a}\,\vec{b}\,\vec{e}][\vec{f}\,\vec{c}\,\vec{d}] - [\vec{a}\,\vec{b}\,\vec{f}][\vec{e}\,\vec{c}\,\vec{d}]$

(3) $[\vec{c}\,\vec{d}\,\vec{a}][\vec{b}\,\vec{e}\,\vec{f}] - [\vec{a}\,\vec{d}\,\vec{b}][\vec{a}\,\vec{e}\,\vec{f}]$

(4) $[\vec{a}\,\vec{c}\,\vec{e}][\vec{b}\,\vec{d}\,\vec{f}]$

30. The scalars l and m such that $l\vec{a} + m\vec{b} = \vec{c}$, where $\vec{a}, \vec{b}$ and $\vec{c}$ are given vectors, are equal to

(1) $l = \dfrac{(\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b})}{(\vec{a}\times\vec{b})^2}$

(2) $l = \dfrac{(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{a})}{(\vec{b}\times\vec{a})}$

(3) $m = \dfrac{(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{a})}{(\vec{b}\times\vec{a})^2}$

(4) $m = \dfrac{(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{a})}{(\vec{b}\times\vec{a})}$

31. If $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\cdot(\vec{a}\times\vec{d}) = 0$, then which of the following may be true?

(1) $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are necessarily coplanar

(2) $\vec{a}$ lies in the plane of $\vec{c}$ and $\vec{d}$

(3) $\vec{b}$ lies in the plane of $\vec{a}$ and $\vec{d}$

(4) $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{d}$

32. A, B, C and D are four points such that $\overrightarrow{AB} = m(2\hat{i} - 6\hat{j} + 2\hat{k})$, $\overrightarrow{BC} = (\hat{i} - 2\hat{j})$ and $\overrightarrow{CD} = n(-6\hat{i} + 15\hat{j} - 3\hat{k})$. If CD intersects AB at some point E, then

(1) $m \geq 1/2$

(2) $n \geq 1/3$

(3) $m = n$

(4) $m < n$

33. If vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar and l, m and n are distinct scalars, then $[(l\vec{a}+m\vec{b}+n\vec{c})(l\vec{b}+m\vec{c}+n\vec{a})(l\vec{c}+m\vec{a}+n\vec{b})]=0$ implies

(1) $l+m+n=0$

(2) roots of the equation $lx^2+mx+n=0$ are real

(3) $l^2+m^2+n^2=0$

(4) $l^3+m^3+n^3=3lmn$

34. Let $\vec{\alpha}=a\hat{i}+b\hat{j}+c\hat{k}$, $\vec{\beta}=b\hat{i}+c\hat{j}+a\hat{k}$ and $\vec{\gamma}=c\hat{i}+a\hat{j}+b\hat{k}$ be three coplanar vectors with $a\neq b$, and $\vec{v}=\hat{i}+\hat{j}+\hat{k}$. Then $\vec{v}$ is perpendicular to

(1) $\vec{\alpha}$ (2) $\vec{\beta}$ (3) $\vec{\gamma}$ (4) none of these

35. If vectors $\vec{A}=2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{B}=\hat{i}+\hat{j}+5\hat{k}$ and $\vec{C}$ form a left-handed system, then $\vec{C}$ is

(1) $11\hat{i}-6\hat{j}-\hat{k}$ (2) $-11\hat{i}+6\hat{j}+\hat{k}$ (3) $11\hat{i}-6\hat{j}+\hat{k}$ (4) $-11\hat{i}+6\hat{j}-\hat{k}$

36. If $\vec{a}=x\hat{i}+y\hat{j}+z\hat{k}, \vec{b}=y\hat{i}+z\hat{j}+x\hat{k}$ and $\vec{c}=z\hat{i}+x\hat{j}+y\hat{k}$, then $\vec{a}\times(\vec{b}\times\vec{c})$ is

(1) parallel to $(y-z)\hat{i}+(z-x)\hat{j}+(x-y)\hat{k}$ (2) orthogonal to $\hat{i}+\hat{j}+\hat{k}$

(3) orthogonal to $(y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$ (4) orthogonal to $x\hat{i}+y\hat{j}+z\hat{k}$

37. If $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$, then

(1) $(\vec{c}\times\vec{a})\times\vec{b}=\vec{0}$ (2) $\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$

(3) $\vec{b}\times(\vec{c}\times\vec{a})=\vec{0}$ (4) $(\vec{c}\times\vec{a})\times\vec{b}=\vec{b}\times(\vec{c}\times\vec{a})=\vec{0}$

38. A vector $(\vec{d})$ is equally inclined to three vectors $\vec{a}=\hat{i}-\hat{j}+\hat{k}, \vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{j}-2\hat{k}$. Let $\vec{x},\vec{y},\vec{z}$ be three vectors in the plane of $\vec{a},\vec{b};\vec{b},\vec{c};\vec{c},\vec{a}$, respectively. Then

(1) $\vec{z}\cdot\vec{d}=0$ (2) $\vec{x}\cdot\vec{d}=1$

(3) $\vec{y}\cdot\vec{d}=32$ (4) $\vec{r}\cdot\vec{d}=0$, where $\vec{r}=\lambda\vec{x}+\mu\vec{y}+\gamma\vec{z}$

39. A parallelogram is constructed on vectors $\vec{a}=3\vec{\alpha}-\vec{\beta}, \vec{b}=\vec{\alpha}+3\vec{\beta}$ if $|\vec{\alpha}|=|\vec{\beta}|=2$, and angle between $\vec{\alpha}$ and $\vec{\beta}$ is $\frac{\pi}{3}$, then the length of a diagonal of parallelogram is

(1) $4\sqrt{5}$ (2) $4\sqrt{3}$ (3) $4\sqrt{7}$ (4) None of these

Linked Comprehension Type

For Problems 1–3

Let $\vec{u},\vec{v}$ and $\vec{w}$ be three unit vectors such that $\vec{u}+\vec{v}+\vec{w}=\vec{a}$, $\vec{u}\times(\vec{v}\times\vec{w})=\vec{b}$, $(\vec{u}\times\vec{v})\times\vec{w}=\vec{c}$, $\vec{a}\cdot\vec{u}=3/2$, $\vec{a}\cdot\vec{v}=7/4$ and $|\vec{a}|=2$.

1. Vector $\vec{u}$ is

(1) $\vec{a}-\frac{2}{3}\vec{b}+\vec{c}$ (2) $\vec{a}+\frac{4}{3}\vec{b}+\frac{8}{3}\vec{c}$ (3) $2\vec{a}-\vec{b}+\frac{1}{3}\vec{c}$ (4) $\frac{4}{3}\vec{a}-\vec{b}+\frac{2}{3}\vec{c}$

2. Vector $\vec{v}$ is

(1) $2\vec{a}-3\vec{c}$ (2) $3\vec{b}-4c$ (3) $-4\vec{c}$ (4) $\vec{a}+\vec{b}+2\vec{c}$

3. Vector $\vec{w}$ is

(1) $\frac{2}{3}(2\vec{c}-\vec{b})$ (2) $\frac{1}{3}(\vec{a}-\vec{b}-\vec{c})$ (3) $\frac{1}{3}\vec{a}-\frac{2}{3}\vec{b}-2\vec{c}$ (4) $\frac{4}{3}(\vec{c}-\vec{b})$

For Problems 4–6

Vectors $\vec{x},\vec{y}$ and $\vec{z}$, each of magnitude $\sqrt{2}$, make an angle of 60° with each other. $\vec{x}\times(\vec{y}\times\vec{z})=\vec{a}$, $\vec{y}\times(\vec{z}\times\vec{x})=\vec{b}$ and $\vec{x}\times\vec{y}=\vec{c}$.

4. Vector $\vec{x}$ is

(1) $\frac{1}{2}[(\vec{a}-\vec{b})\times\vec{c}+(\vec{a}+\vec{b})]$ (2) $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+(\vec{a}-\vec{b})]$

(3) $\frac{1}{2}[-(\vec{a}+\vec{b})\times\vec{c}+(\vec{a}+\vec{b})]$ (4) $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}-(\vec{a}+\vec{b})]$

5. Vector $\vec{y}$ is

(1) $\frac{1}{2}[(\vec{a}+\vec{c})\times\vec{b}-\vec{b}-\vec{a}]$ (2) $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{c}+\vec{b}+\vec{a}]$

(3) $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}+\vec{a}]$ (4) $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{a}+\vec{b}-\vec{a}]$

6. Vector $\vec{z}$ is

(1) $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{c}-\vec{b}+\vec{a}]$ (2) $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}-\vec{a}]$

(3) $\frac{1}{2}[\vec{c}\times(\vec{a}-\vec{b})+\vec{b}+\vec{a}]$ (4) none of these

For Problems 7–9

If $\vec{x}\times\vec{y}=\vec{a}$, $\vec{y}\times\vec{z}=\vec{b}$, $\vec{x}\cdot\vec{b}=\gamma$, $\vec{x}\cdot\vec{y}=1$ and $\vec{y}\cdot\vec{z}=1$

7. Vector $\vec{x}$ is

(1) $\frac{1}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times(\vec{a}\times\vec{b})]$ (2) $\frac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}-\vec{a}\times(\vec{a}\times\vec{b})]$

(3) $\frac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}+\vec{b}\times(\vec{a}\times\vec{b})]$ (4) none of these

8. Vector $\vec{y}$ is

(1) $\dfrac{\vec{a}\times\vec{b}}{\gamma}$ (2) $\vec{a}+\dfrac{\vec{a}\times\vec{b}}{\gamma}$ (3) $\vec{a}+\vec{b}+\dfrac{\vec{a}\times\vec{b}}{\gamma}$ (4) none of these

9. Vector $\vec{z}$ is

(1) $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}+\vec{b}\times(\vec{a}\times\vec{b})]$ (2) $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}+\vec{b}-\vec{a}\times(\vec{a}\times\vec{b})]$

(3) $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}+\vec{b}\times(\vec{a}\times\vec{b})]$ (4) none of these

For Problems 10–12

Given two orthogonal vectors $\vec{A}$ and $\vec{B}$ each of length unity. Let $\vec{P}$ be the vector satisfying the equation $\vec{P}\times\vec{B}=\vec{A}-\vec{P}$. Then

10. $(\vec{P}\times\vec{B})\times\vec{B}$ is equal to

(1) $\vec{P}$ (2) $-\vec{P}$ (3) $2\vec{B}$ (4) $\vec{A}$

11. $\vec{P}$ is equal to

(1) $\dfrac{\vec{A}}{2}+\dfrac{\vec{A}\times\vec{B}}{2}$ (2) $\dfrac{\vec{A}}{2}+\dfrac{\vec{B}\times\vec{A}}{2}$ (3) $\dfrac{\vec{A}\times\vec{B}}{2}-\dfrac{\vec{A}}{2}$ (4) $\vec{A}\times\vec{B}$

12. Which of the following statements is false?

(1) vectors $\vec{P}, \vec{A}$ and $\vec{P}\times\vec{B}$ are linearly dependent.

(2) vectors $\vec{P}, \vec{B}$ and $\vec{P}\times\vec{B}$ are linearly independent.

(3) $\vec{P}$ is orthogonal to $\vec{B}$ and has length $1/\sqrt{2}$.

(4) none of the above.

For Problems 13–15

Let $\vec{a}=2\hat{i}+3\hat{j}-6\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{c}=-2\hat{i}+3\hat{j}+6\hat{k}$. Let $\vec{a_1}$ be the projection of $\vec{a}$ on $\vec{b}$ and $\vec{a_2}$ be the projection of $\vec{a_1}$ on $\vec{c}$. Then

13. $\vec{a_2}$ is equal to

(1) $\dfrac{943}{49}(2\hat{i}-3\hat{j}-6\hat{k})$ (2) $\dfrac{943}{49^2}(2\hat{i}-3\hat{j}-6\hat{k})$

(3) $\dfrac{943}{49}(-2\hat{i}+3\hat{j}+6\hat{k})$ (4) $\dfrac{943}{49^2}(-2\hat{i}+3\hat{j}+6\hat{k})$

14. $\vec{a_1} \cdot \vec{b}$ is equal to
(1) -41 (2) $-41/7$ (3) 41 (4) 287

15. Which of the following is true?
(1) $\vec{a}$ and $\vec{a_2}$ are collinear
(2) $\vec{a_1}$ and $\vec{c}$ are collinear
(3) $\vec{a}, \vec{a_1}$ and $\vec{b}$ are coplanar
(4) $\vec{a}, \vec{a_1}$ and $\vec{a_2}$ are coplanar

For Problems 16–18

Consider a triangular pyramid $ABCD$ the position vectors of whose angular points are $A(3, 0, 1)$, $B(-1, 4, 1)$, $C(5, 2, 3)$ and $D(0, -5, 4)$. Let G be the point of intersection of the medians of triangle BCD.

16. The length of vector $\overrightarrow{AG}$ is
(1) $\sqrt{17}$ (2) $\sqrt{51}/3$ (3) $3/\sqrt{6}$ (4) $\sqrt{59}/4$

17. Area of triangle ABC in sq. units is
(1) 24 (2) $8\sqrt{6}$ (3) $4\sqrt{6}$ (4) none of these

18. The length of the perpendicular from vertex D on the opposite face is
(1) $14/\sqrt{6}$ (2) $2/\sqrt{6}$ (3) $3/\sqrt{6}$ (4) none of these

For Problems 19–21

Vertices of a parallelogram taken in order are $A(2, -1, 4)$; $B(1, 0 -1)$; $C(1, 2, 3)$ and D.

19. The distance between the parallel lines AB and CD is
(1) $\sqrt{6}$ (2) $3\sqrt{6/5}$ (3) $2\sqrt{2}$ (4) 3

20. Distance of the point $P(8, 2, -12)$ from the plane of the parallelogram is
(1) $\frac{4\sqrt{6}}{9}$ (2) $\frac{32\sqrt{6}}{9}$ (3) $\frac{16\sqrt{6}}{9}$ (4) none of these

21. The orthogonal projections of the parallelogram on the three coordinate planes xy, yz and zx, respectively, are
(1) 14, 4, 2 (2) 2, 4, 14 (3) 4, 2, 14 (4) 2, 14, 4

For Problems 22–24

Let $\vec{r}$ be a position vector of a variable point in Cartesian OXY plane such that $\vec{r} \cdot (10\hat{j} - 8\hat{i} - \vec{r}) = 40$ and $p_1 = \max\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$, $p_2 = \min\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$. A tangent line is drawn to the curve $y = 8/x^2$ at point A with abscissa 2. The drawn line cuts the x-axis at a point B.

22. p_2 is equal to
(1) 9 (2) $2\sqrt{2} - 1$ (3) $6\sqrt{2} + 3$ (4) $9 - 4\sqrt{2}$

23. $p_1 + p_2$ is equal to
(1) 2 (2) 10 (3) 18 (4) 5

24. $\overrightarrow{AB} \cdot \overrightarrow{OB}$ is equal to

(1) 1 (2) 2 (3) 3 (4) 4

For Problems 25–27

AB, AC and AD are three adjacent edges of a parallelepiped. The diagonal of the parallelepiped passing through A and directed away from it is vector $\vec{a}$. The vector area of the faces containing vertices A, B, C and A, B, D are $\vec{b}$ and $\vec{c}$, respectively, i.e., $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{b}$ and $\overrightarrow{AD} \times \overrightarrow{AB} = \vec{c}$. The projection of each edge AB and AC on diagonal vector $\vec{a}$ is $\frac{|\vec{a}|}{3}$.

25. Vector $\overrightarrow{AB}$ is

(1) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2}$ (2) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2} + \frac{3(\vec{b} \times \vec{a})}{|\vec{a}|^2}$

(3) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2} - \frac{3(\vec{c} \times \vec{a})}{|\vec{a}|^2}$ (4) none of these

26. Vector $\overrightarrow{AC}$ is

(1) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2}$ (2) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2} + \frac{3(\vec{b} \times \vec{a})}{|\vec{a}|^2}$

(3) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2} - \frac{3(\vec{c} \times \vec{a})}{|\vec{a}|^2}$ (4) none of these

27. Vector $\overrightarrow{AD}$ is

(1) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2}$ (2) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2} + \frac{3(\vec{b} \times \vec{a})}{|\vec{a}|^2}$

(3) $\frac{1}{3}\vec{a} + \frac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2} - \frac{3(\vec{c} \times \vec{a})}{|\vec{a}|^2}$ (4) none of these

Matrix Match Type

1. Match the following lists:

List I	List II
a. The possible value of a if $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{r} = (\hat{i} + 2\hat{j}) + \mu(-\hat{i} + \hat{j} + a\hat{k})$ are not consistent, where λ and μ are scalars, is	**p.** -4
b. The angle between vectors $\vec{a} = \lambda\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2\lambda\hat{i} + \lambda\hat{j} - \hat{k}$ is acute, whereas vector $\vec{b}$ makes an obtuse angle with the axes of coordinates. Then λ may be	**q.** -2

c. The possible value of a such that $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}+(1+a)\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar is	r. 2
d. If $\vec{A}=2\hat{i}+\lambda\hat{j}+3\hat{k}$, $\vec{B}=2\hat{i}+\lambda\hat{j}+\hat{k}$, $\vec{C}=3\hat{i}+\hat{j}$ and $\vec{A}+\lambda\vec{B}$ is perpendicular to $\vec{C}$, then $\lvert 2\lambda\rvert$ is	s. 3

2. Match the following lists:

List I	List II
a. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors where $\lvert\vec{a}\rvert=\lvert\vec{b}\rvert=2, \lvert\vec{c}\rvert=1$, then $[\vec{a}\times\vec{b}\quad\vec{b}\times\vec{c}\quad\vec{c}\times\vec{a}]$ is	p. -12
b. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at $\pi/3$, then $16[\vec{a}\quad\vec{b}+\vec{a}\times\vec{b}\quad\vec{b}]$ is	q. 0
c. If $\vec{b}$ and $\vec{c}$ are orthogonal unit vectors and $\vec{b}\times\vec{c}=\vec{a}$, then $[\vec{a}+\vec{b}+\vec{c}\quad\vec{a}+\vec{b}\quad\vec{b}+\vec{c}]$ is	r. 16
d. If $[\vec{x}\ \vec{y}\ \vec{a}]=[\vec{x}\ \vec{y}\ \vec{b}]=[\vec{a}\ \vec{b}\ \vec{c}]=0$, each vector being a non-zero vector, then $[\vec{x}\ \vec{y}\ \vec{c}]$ is	s. 1

3. Match the following lists:

List I	List II
a. If $\lvert\vec{a}\rvert=\lvert\vec{b}\rvert=\lvert\vec{c}\rvert$, angle between each pair of vectors is $\frac{\pi}{3}$ and $\lvert\vec{a}+\vec{b}+\vec{c}\rvert=\sqrt{6}$, then $2\lvert\vec{a}\rvert$ is equal to	p. 3
b. If $\vec{a}$ is perpendicular to $\vec{b}+\vec{c}$, $\vec{b}$ is perpendicular to $\vec{c}+\vec{a}$, $\vec{c}$ is perpendicular to $\vec{a}+\vec{b}$, $\lvert\vec{a}\rvert=2$, $\lvert\vec{b}\rvert=3$ and $\lvert\vec{c}\rvert=6$, then $\lvert\vec{a}+\vec{b}+\vec{c}\rvert-2$ is equal to	q. 2
c. $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}-4\hat{k}$, $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{d}=3\hat{i}+2\hat{j}+\hat{k}$, then $\frac{1}{7}(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$ is equal to	r. 4
d. If $\lvert\vec{a}\rvert=\lvert\vec{b}\rvert=\lvert\vec{c}\rvert=2$ and $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=2$, then $[\vec{a}\ \vec{b}\ \vec{c}]\cos 45°$ is equal to	s. 5

4. Given two vectors $\vec{a} = -\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{j} - \hat{k}$. Now, match the following lists:

List I	List II
a. Area of triangle formed by $\vec{a}$ and $\vec{b}$	p. 3
b. Area of parallelogram having sides $\vec{a}$ and $\vec{b}$	q. $12\sqrt{3}$
c. Area of parallelogram having diagonals $2\vec{a}$ and $4\vec{b}$	r. $3\sqrt{3}$
d. Volume of parallelepiped formed by $\vec{a}, \vec{b}$ and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$	s. $\frac{3\sqrt{3}}{2}$

5. Given two vectors $\vec{a} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$. Now, match the following lists:

List I	List II
a. A vector coplanar with $\vec{a}$ and $\vec{b}$	p. $-3\hat{i} + 3\hat{j} + 4\hat{k}$
b. A vector which is perpendicular to both $\vec{a}$ and $\vec{b}$	q. $2\hat{i} - 2\hat{j} + 3\hat{k}$
c. A vector which is equally inclined to $\vec{a}$ and $\vec{b}$	r. $\hat{i} + \hat{j}$
d. A vector which forms a triangle with $\vec{a}$ and $\vec{b}$	s. $\hat{i} - \hat{j} + 5\hat{k}$

6. Match the following lists:

List I	List II
a. If $\lvert\vec{a} + \vec{b}\rvert = \lvert\vec{a} + 2\vec{b}\rvert$, then angle between $\vec{a}$ and $\vec{b}$ is	p. 90°
b. If $\lvert\vec{a} + \vec{b}\rvert = \lvert\vec{a} - 2\vec{b}\rvert$, then angle between $\vec{a}$ and $\vec{b}$ is	q. obtuse
c. If $\lvert\vec{a} + \vec{b}\rvert = \lvert\vec{a} - \vec{b}\rvert$, then angle between $\vec{a}$ and $\vec{b}$ is	r. 0°
d. Angle between $\vec{a} \times \vec{b}$ and a vector perpendicular to the vector $\vec{c} \times (\vec{a} \times \vec{b})$ is	s. acute

7. Volume of parallelepiped formed by vectors $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ is 36 sq. units. Now, match the following lists:

List I	List II
a. Volume of parallelepiped formed by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is	p. 0 sq. units
b. Volume of tetrahedron formed by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is	q. 12 sq. units
c. Volume of parallelepiped formed by vectors $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ is	r. 6 sq. units
d. Volume of parallelepiped formed by vectors $\vec{a}-\vec{b}, \vec{b}-\vec{c}$ and $\vec{c}-\vec{a}$ is	s. 1 sq. units

Numerical Value Type

1. If $\vec{a}$ and $\vec{b}$ are any two unit vectors, then find the greatest positive integer in the range of $\frac{3|\vec{a}+\vec{b}|}{2} + 2|\vec{a}-\vec{b}|$.

2. Let $\vec{u}$ be a vector on rectangular coordinate system with sloping angle 60°. Suppose that $|\vec{u}-\hat{i}|$ is geometric mean of $|\vec{u}|$ and $|\vec{u}-2\hat{i}|$, where $\hat{i}$ is the unit vector along the x-axis. Then find the value of $(\sqrt{2}+1)|\vec{u}|$.

3. Find the absolute value of parameter t for which the area of the triangle whose vertices are $A(-1, 1, 2)$; $B(1, 2, 3)$ and $C(t, 1, 1)$ is minimum.

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ and $[3\vec{a}+\vec{b}\ 3\vec{b}+\vec{c}\ 3\vec{c}+\vec{a}]$
$= \lambda \begin{vmatrix} \vec{a}\cdot\hat{i} & \vec{a}\cdot\hat{j} & \vec{a}\cdot\hat{k} \\ \vec{b}\cdot\hat{i} & \vec{b}\cdot\hat{j} & \vec{b}\cdot\hat{k} \\ \vec{c}\cdot\hat{i} & \vec{c}\cdot\hat{j} & \vec{c}\cdot\hat{k} \end{vmatrix}$, then find the value of $\frac{\lambda}{4}$.

5. Let $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$. Find the value of 6α, such that $\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\times(\vec{c}\times\vec{a}) = 0$.

6. If $\vec{x}, \vec{y}$ are two non-zero and non-collinear vectors satisfying
$[(a-2)\alpha^2 + (b-3)\alpha + c]\vec{x} + [(a-2)\beta^2 + (b-3)\beta + c]\vec{y} + [(a-2)\gamma^2 + (b-3)\gamma + c](\vec{x}\times\vec{y}) = 0$,
where α, β, γ are three distinct real numbers, then find the value of $(a^2 + b^2 + c^2 - 4)$.

7. Let $\vec{u}$ and $\vec{v}$ be unit vectors such that $\vec{u}\times\vec{v} + \vec{u} = \vec{w}$ and $\vec{w}\times\vec{u} = \vec{v}$. Find the value of $[\vec{u}\ \vec{v}\ \vec{w}]$.

8. Find the value of λ if the volume of a tetrahedron whose vertices are with position vectors $\hat{i} - 6\hat{j} + 10\hat{k}, -\hat{i} - 3\hat{j} + 7\hat{k}, 5\hat{i} - \hat{j} + \lambda\hat{k}$ and $7\hat{i} - 4\hat{j} + 7\hat{k}$ is 11 cubic unit.

9. Given that $\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}$; $\vec{w} = \hat{i} + 3\hat{j} + 3\hat{k}$ and $(\vec{u}\cdot\vec{R} - 15)\hat{i} + (\vec{v}\cdot\vec{R} - 30)\hat{j} + (\vec{w}\cdot\vec{R} - 20)\hat{k} = \vec{0}$. Then find the greatest integer less than or equal to $|\vec{R}|$.

10. Let a three-dimensional vector $\vec{V}$ satisfy the condition, $2\vec{V} + \vec{V} \times (\hat{i} + 2\hat{j}) = 2\hat{i} + \hat{k}$. If $3|\vec{V}| = \sqrt{m}$, then find the value of m.

11. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = 0 = \vec{a}\cdot\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$, then find the value of $|\vec{a}\times\vec{b} - \vec{a}\times\vec{c}|$.

12. Let $\vec{OA} = \vec{a}, \vec{OB} = 10\vec{a} + 2\vec{b}$ and $\vec{OC} = \vec{b}$, where O, A and C are non-collinear points. Let p denote the area of quadrilateral $OACB$, and let q denote the area of parallelogram with OA and OC as adjacent sides. If $p = k\,q$, then find k.

13. Find the work done by the force $F = 3\hat{i} - \hat{j} - 2\hat{k}$ acting on a particle such that the particle is displaced from point A $(-3, -4, 1)$ to point B $(-1, -1, -2)$.

Archives

JEE Main

Single Correct Answer Type

1. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $[3\vec{u}\;p\vec{v}\;p\vec{w}] - [p\vec{v}\;\vec{w}\;q\vec{u}] - [2\vec{w}\;q\vec{v}\;q\vec{u}] = 0$ holds for
(1) exactly one value of (p, q).
(2) exactly two values of (p, q).
(3) more than two but not all values of (p, q).
(4) all values of (p, q). **(AIEEE 2009)**

2. Let $\vec{a} = \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$. Then the vector $\vec{b}$ satisfying $\vec{a}\times\vec{b} + \vec{c} = 0$ and $\vec{a}\cdot\vec{b} = 3$ is
(1) $\hat{i} + \hat{j} - 2\hat{k}$ (2) $-\hat{i} + \hat{j} - 2\hat{k}$ (3) $2\hat{i} - \hat{j} + 2\hat{k}$ (4) $\hat{i} - \hat{j} - 2\hat{k}$ **(AIEEE 2010)**

3. If the vectors $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$, and $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ are mutually orthogonal, then (λ, μ) is equal to
(1) $(3, -2)$ (2) $(-3, 2)$ (3) $(2, -3)$ (4) $(-2, 3)$ **(AIEEE 2010)**

4. If $\vec{a} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$ and $\vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$, then the value of $(2\vec{a} - \vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a} + 2\vec{b})]$ is
(1) 3 (2) -5 (3) -3 (4) 5 **(AIEEE 2011)**

5. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying $\vec{b}\times\vec{c} = \vec{b}\times\vec{d}$ and $\vec{a}\cdot\vec{d} = 0$. Then the vector $\vec{d}$ is equal to
(1) $\vec{c} - \left(\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{b}$ (2) $\vec{b} - \left(\frac{\vec{b}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{c}$ (3) $\vec{c} + \left(\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{b}$ (4) $\vec{b} + \left(\frac{\vec{b}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{c}$ **(AIEEE 2011)**

6. Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\vec{c} = \hat{a} + 2\hat{b}$ and $\vec{d} = 5\hat{a} - 4\hat{b}$ are perpendicular to each other, then the angle between $\hat{a}$ and $\hat{b}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

(AIEEE 2012)

7. Let $ABCD$ be a parallelogram such that $\overrightarrow{AB} = \vec{q}, \overrightarrow{AD} = \vec{p}$ and $\angle BAD$ be an acute angle. If $\vec{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then $\vec{r}$ is given by

(1) $\vec{r} = 3\vec{q} - \frac{3(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}\vec{p}$ (2) $\vec{r} = -\vec{q} + \left(\frac{\vec{p}\cdot\vec{q}}{\vec{p}\cdot\vec{p}}\right)\vec{p}$

(3) $\vec{r} = \vec{q} - \left(\frac{\vec{p}\cdot\vec{q}}{\vec{p}\cdot\vec{p}}\right)\vec{p}$ (4) $\vec{r} = -3\vec{q} + \frac{3(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}$ **(AIEEE 2012)**

8. If $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = \lambda[\vec{a}\,\vec{b}\,\vec{c}]^2$, then λ is equal to

(1) 2 (2) 3 (3) 0 (4) 1 **(JEE Main 2014)**

9. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a}\times\vec{b})\times\vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\,\vec{a}$. If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ is

(1) $\frac{2\sqrt{2}}{3}$ (2) $\frac{-2}{3}$ (3) $\frac{2}{3}$ (4) $\frac{-2\sqrt{3}}{3}$

(JEE Main 2015)

10. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{2\pi}{3}$ (3) $\frac{5\pi}{6}$ (4) $\frac{3\pi}{4}$

(JEE Main 2016)

11. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c} - \vec{a}| = 3, |(\vec{a}\times\vec{b})\times\vec{c}| = 3$ and the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$ be 30°. Then $\vec{a}\cdot\vec{c}$ is equal to

(1) $\frac{1}{8}$ (2) $\frac{25}{8}$ (3) 2 (4) 5

(JEE Main 2017)

12. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}\cdot\vec{b} = 24$, then $|\vec{u}|^2$ is equal to

(1) 84 (2) 336 (3) 315 (4) 256

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 1$ and $\vec{a} \cdot \vec{c} = \frac{1}{2}$, then

(1) $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar

(2) $\vec{b}, \vec{c}$ and $\vec{d}$ are non-coplanar

(3) $\vec{b}$ and $\vec{d}$ are non-parallel

(4) $\vec{a}$ and $\vec{d}$ are parallel and $\vec{b}$ and $\vec{c}$ are parallel **(IIT-JEE, 2009)**

2. Two adjacent sides of a parallelogram $ABCD$ are given by $\overrightarrow{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by

(1) $\frac{8}{9}$ (2) $\frac{\sqrt{17}}{9}$ (3) $\frac{1}{9}$ (4) $\frac{4\sqrt{5}}{9}$

(IIT-JEE, 2010)

3. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}, 4\hat{i}, 3\hat{i} + 3\hat{j}$ and $-3\hat{j} + 2\hat{j}$, respectively. The quadrilateral $PQRS$ must be a

(1) parallelogram, which is neither a rhombus nor a rectangle

(2) square

(3) rectangle, but not a square

(4) rhombus, but not a square **(IIT-JEE, 2010)**

4. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

(1) $\hat{i} - 3\hat{j} + 3\hat{k}$ (2) $-3\hat{i} - 3\hat{j} + \hat{k}$ (3) $3\hat{i} - \hat{j} + 3\hat{k}$ (4) $\hat{i} + 3\hat{j} - 3\hat{k}$

(IIT-JEE, 2011)

5. Let $\overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ determine diagonals of a parallelogram $PQRS$, and $\overrightarrow{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}, \overrightarrow{PQ}$ and $\overrightarrow{PS}$ is

(1) 5 (2) 20 (3) 10 (4) 30

(JEE Advanced 2013)

6. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that $\overrightarrow{OP} \cdot \overrightarrow{OQ} + \overrightarrow{OR} \cdot \overrightarrow{OS} = \overrightarrow{OR} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OS} = \overrightarrow{OQ} \cdot \overrightarrow{OR} + \overrightarrow{OP} \cdot \overrightarrow{OS}$ then the triangle PQR has S as its

(1) centroid (2) circumcentre (3) incentre (4) orthocenter

(JEE Advanced 2017)

Multiple Correct Answers Type

1. The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to vector $\hat{i}+\hat{j}+\hat{k}$, is/are

(1) $\hat{j}-\hat{k}$ (2) $-\hat{i}+\hat{j}$ (3) $\hat{i}-\hat{j}$ (4) $-\hat{j}+\hat{k}$

(IIT-JEE, 2011)

2. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$ and $\vec{b}$ is a non-zero vector perpendicular to $\vec{y}$ and $\vec{z}\times\vec{x}$, then

(1) $\vec{b}=(\vec{b}\cdot\vec{z})(\vec{z}-\vec{x})$ (2) $\vec{a}=(\vec{a}\cdot\vec{y})(\vec{y}-\vec{z})$

(3) $\vec{a}\cdot\vec{b}=-(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})$ (4) $\vec{a}=(\vec{a}\cdot\vec{y})(\vec{z}-\vec{y})$ **(JEE Advanced 2014)**

3. Let ΔPQR be a triangle. Let $\vec{a}=\overrightarrow{QR}$, $\vec{b}=\overrightarrow{RP}$ and $\vec{c}=\overrightarrow{PQ}$. If $|\vec{a}|=12, |\vec{b}|=4\sqrt{3}$ and $\vec{b}\cdot\vec{c}=24$, then which of the following is (are) true?

(1) $\frac{|\vec{c}|^2}{2}-|\vec{a}|=12$ (2) $\frac{|\vec{c}|^2}{2}+|\vec{a}|=30$

(3) $|\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|=48\sqrt{3}$ (4) $\vec{a}\cdot\vec{b}=-72$ **(JEE Advanced 2015)**

4. Let $\hat{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ be a unit vector in R^3 and $\hat{\omega}=\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$. Given that there exists a vector $\vec{v}$ in R^3 such that $|\hat{u}\times\vec{v}|=1$ and $\hat{\omega}.(\hat{u}\times\vec{v})=1$. Which of the following statement(s) is (are) correct ?

(1) There is exactly one choice for such $\vec{v}$

(2) There are infinitely many choices for such $\vec{v}$

(3) If $\hat{u}$ lies in the xy-plane then $|u_1|=|u_2|$

(4) If $\hat{u}$ lies in the xz-plane then $2|u_1|=|u_3|$ **(JEE Advanced 2016)**

Linked Comprehension Type

For Problems 1 and 2

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR. **(JEE Advanced 2017)**

1. $\left|\overrightarrow{OX}\times\overrightarrow{OY}\right|=$

(1) $\sin(P+Q)$ (2) $\sin 2R$ (3) $\sin(P+R)$ (4) $\sin(Q+R)$

2. If the triangle PQR varies, then the minimum value of $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$ is

(1) $-\frac{5}{3}$ (2) $-\frac{3}{2}$ (3) $\frac{3}{2}$ (4) $\frac{5}{3}$

Matrix Match Type

1. Match the following lists:

List I	List II
a. Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	**p.** 100
b. Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}), (\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	**q.** 30
c. Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	**r.** 24
d. Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	**s.** 60

(JEE Advanced 2013)

2. Match the statements given in List I with the values given in List II.

List I	List II
a. Let $y(x)=\cos(3\cos^{-1}x), x\in[-1,1], x\neq\pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals	**p.** 1
b. Let $A_1, A_2, \ldots, A_n$ $(n>2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a_k}$ be the position vector of the point A_k, $k=1, 2, \ldots n$. If $\left\lvert\sum_{k=1}^{n-1}(\vec{a_k}\times\vec{a_{k+1}})\right\rvert = \left\lvert\sum_{k=1}^{n-1}(\vec{a_k}\cdot\vec{a_{k+1}})\right\rvert$, then the minimum value of n is	**q.** 2
c. If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line $x+y=8$, then the value of h is	**r.** 8
d. Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is	**s.** 9

Codes:	a	b	c	d
(1)	s	r	q	p
(2)	q	s	r	p
(3)	s	r	p	q
(4)	q	s	p	r

(JEE Advanced 2014)

3. Match the following lists:

List I	List II
a. In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha=2+\sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	p. 1
b. Let a and b be real numbers such that the function $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\geq 1\end{cases}$ is differentiable for all $x\in R$. Then possible value(s) of α is (are)	q. 2
c. Let $\omega\neq 1$ be a complex cube root of unity If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$, then possible value(s) of n is (are)	r. 3
d. Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is an arithmetic progression, then the value(s) of $\lvert q-a\rvert$ is (are)	s. 4
	t. 5

(JEE Advanced 2015)

Numerical Value Type

1. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a}=\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then find the value of $(2\vec{a}+\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})]$.

(IIT-JEE, 2010)

2. Let $\vec{a}=-\hat{i}-\hat{k}, \vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{r}\cdot\vec{a}=0$, then find the value of $\vec{r}\cdot\vec{b}$.

(IIT-JEE, 2011)

3. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $\lvert\vec{a}-\vec{b}\rvert^2+\lvert\vec{b}-\vec{c}\rvert^2+\lvert\vec{c}-\vec{a}\rvert^2=9$, then $\lvert 2\vec{a}+5\vec{b}+5\vec{c}\rvert$ is _____.

(IIT-JEE, 2012

4. Let $\vec{a}, \vec{b}$, and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$, where p, q and r are scalars, then the value of $\frac{p^2+2q^2+r^2}{q^2}$ is _____.

(JEE Advanced 2014)

5. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a} \cdot \vec{b} = 0$. For some $x, y \in R$, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is _____.

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	2. (4)	3. (3)	4. (2)	5. (3)	6. (4)	7. (3)	8. (2)	9. (3)	10. (2)
11. (3)	12. (3)	13. (4)	14. (3)	15. (1)	16. (1)	17. (4)	18. (3)	19. (4)	20. (2)
21. (1)	22. (1)	23. (1)	24. (1)	25. (2)	26. (1)	27. (3)	28. (3)	29. (2)	30. (3)
31. (3)	32. (2)	33. (1)	34. (2)	35. (2)	36. (4)	37. (3)	38. (3)	39. (2)	40. (1)
41. (4)	42. (3)	43. (4)	44. (3)	45. (3)	46. (4)	47. (3)	48. (2)	49. (3)	50. (1)
51. (2)	52. (1)	53. (4)	54. (1)	55. (4)	56. (3)	57. (1)	58. (4)	59. (3)	60. (2)
61. (1)	62. (4)	63. (3)	64. (3)	65. (2)	66. (3)	67. (1)	68. (1)	69. (1)	70. (4)
71. (4)	72. (1)	73. (2)	74. (3)	75. (3)	76. (1)	77. (1)	78. (1)	79. (3)	80. (2)
81. (3)	82. (2)	83. (1)	84. (2)	85. (2)	86. (2)	87. (1)	88. (2)	89. (3)	90. (3)
91. (3)	92. (4)	93. (1)	94. (4)	95. (4)					

Multiple Correct Answers Type

1. (1), (2)	2. (1), (3)	3. (1), (2), (3), (4)	4. (1), (2), (3)	5. (1), (3)
6. (1), (3)	7. (1), (2), (3), (4)	8. (1), (2), (3), (4)	9. (1), (2), (3), (4)	10. (2), (4)
11. (2), (4)	12. (2), (4)	13. (1), (4)	14. (2), (3), (4)	15. (1), (4)
16. (2), (4)	17. (2), (3)	18. (3), (4)	19. (2), (4)	20. (2), (4)
21. (1), (2), (3)	22. (1), (2)	23. (1), (2), (3)	24. (1), (2)	25. (1), (3)
26. (2), (4)	27. (2), (3)	28. (1), (2), (3)	29. (1), (2), (3)	30. (1), (3)
31. (2), (3), (4)	32. (1), (2)	33. (1), (2), (4)	34. (1), (2), (3)	35. (2), (4)
36. (1), (2), (3), (4)	37. (1), (3), (4)	38. (1), (4)	39. (2), (3)	

Linked Comprehension Type

1. (2)	2. (3)	3. (4)	4. (4)	5. (3)	6. (2)	7. (2)	8. (1)	9. (3)	10. (2)
11. (2)	12. (4)	13. (2)	14. (1)	15. (3)	16. (2)	17. (3)	18. (1)	19. (3)	20. (2)
21. (4)	22. (4)	23. (3)	24. (3)	25. (1)	26. (2)	27. (3)			

Matrix Match Type

1. a → p, q, r, s; b → q, q; c → p, r; d → r
2. a → r; b → p; c → s; d → q.
3. a → q; b → s; c → p; d → r
4. a → s; b → r; c → q; d → p
5. a → p, r; b → q; c → p, q, s; d → p
6. a → q; b → s; c → p; d → r
7. a → r; b → s; c → q; d → p

Numerical Value Type

1. (5)	2. (1)	3. (2)	4. (7)	5. (4)	6. (9)	7. (1)	8. (7)	9. (6)	10. (6)
11. (1)	12. (6)	13. (9)							

ARCHIVES

JEE MAIN

Single Correct Answer Type

1. (1) **2.** (2) **3.** (2) **4.** (2) **5.** (1) **6.** (3) **7.** (2) **8.** (4) **9.** (1) **10.** (3)
11. (3) **12.** (2)

JEE ADVANCED

Single Correct Answer Type

1. (3) **2.** (2) **3.** (1) **4.** (3) **5.** (3) **6.** (4)

Multiple Correct Answers Type

1. (1), (4) **2.** (1), (2), (3) **3.** (1), (3), (4) **4.** (2), (3)

Linked Comprehension Type

1. (1) **2.** (2)

Matrix Match Type

1. $a \rightarrow r; b \rightarrow s; c \rightarrow p; d \rightarrow q$ **2.** (1)
3. $a \rightarrow p; q$

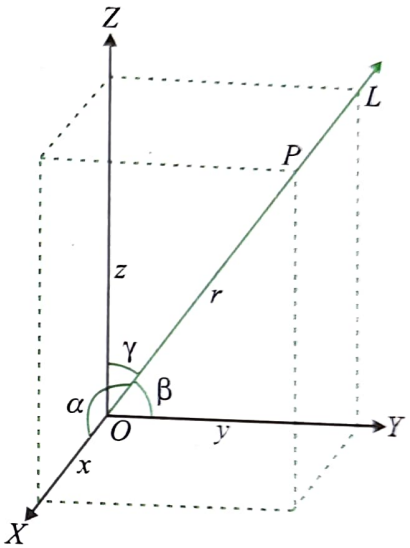
Numerical Value Type

1. (5) **2.** (9) **3.** (3) **4.** (4) **5.** (3)

4 Line and Plane

DIRECTION COSINES AND DIRECTION RATIOS

From Chapter 1, recall that if a directed line L, passing through the origin, makes angles α, β and γ with the x-, y- and z-axes, respectively, called direction angles, then the cosines of these angles, namely, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines of the directed line L.



If we reverse the direction of L, the direction angles are replaced by their supplements, i.e., $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Thus, the signs of the direction cosines are reversed.

Note that a given line in space can be extended in two opposite directions, and so it has two sets of direction cosines. In order to have a unique set of direction cosines for a given line in space, we must take the given line as a directed line. These unique direction cosines are denoted by l, m and n.

If the given line in space does not pass through the origin, then in order to find its direction cosines, we draw a line through the origin and parallel to the given line. Now take one of the directed lines from the origin and find its direction cosines as two parallel lines have the same set of direction cosines.

Any three numbers which are proportional to the direction cosines of a line are called the *direction ratios* of the line. If l, m and n are direction cosines and a, b and c are the direction ratios of a line, then $a = \lambda l$, $b = \lambda m$ and $c = \lambda n$ for any non-zero $\lambda \in R$.

Note:

- Direction cosines of the x-axis are $(1, 0, 0)$.
  Direction cosines of the y-axis are $(0, 1, 0)$.
  Direction cosines of the z-axis are $(0, 0, 1)$.
- Let OP be any line passing through the origin O which has direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, i.e., (l, m, n) where distance $OP = r$, i.e., coordinates of P are $(r\cos\alpha, r\cos\beta, r\cos\gamma)$.
- If l, m and n are the direction cosines of a vector, then $l^2 + m^2 + n^2 = 1$.
- $\vec{r} = |\vec{r}|\,(l\hat{i} + m\hat{j} + n\hat{k})$ and $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$.

DIRECTION RATIOS

Let l, m and n be the direction cosines of a vector $\vec{r}$ and a, b and c be three numbers such that a, b, c are proportional to l, m and n. Therefore,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ or } (l, m, n) = (ka, kb, kc)$$

Hence, a, b and c are direction ratios.

For example, if $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ are direction cosines of a vector $\vec{r}$, then its direction ratios are $(1, -1, 1)$ or $(-1, 1, -1)$ or $(2, -2, 2)$ or $(\lambda, -\lambda, \lambda)$ …

It is evident from the above definition that to obtain the direction ratios of a vector from its direction cosines, we just multiply them by a common number.

"That shows there can be an infinite number of direction ratios for a given vector, but the direction cosines are unique."

Direction Ratios of a Line Joining Two Points

For points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$,

$$\text{Vector } \overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Then the direction ratios of PQ are $\langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle$.

To Obtain Direction Cosines from Direction Ratios

Let a, b and c be the direction ratios of a vector $\vec{r}$ having direction cosines l, m and n. Then

$$l = \lambda a,\ m = \lambda b,\ n = \lambda c \qquad \text{(by definition)}$$

$$\therefore \quad l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \quad a^2\lambda^2 + b^2\lambda^2 + c^2\lambda^2 = 1$$

$$\Rightarrow \quad \lambda = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\Rightarrow \quad l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Example:

Let the direction ratios of a line be 3, 1 and –2.

Direction cosines are

$$\left(\frac{3}{\sqrt{3^2 + 1^2 + (-2)^2}}, \frac{1}{\sqrt{3^2 + 1^2 + (-2)^2}}, \frac{-2}{\sqrt{3^2 + 1^2 + (-2)^2}}\right) \Rightarrow \left(\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$$

Note:

- If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ is a vector having direction cosines l, m and n, then $l = \frac{a}{|\vec{r}|}$, $m = \frac{b}{|\vec{r}|}$, $n = \frac{c}{|\vec{r}|}$.
- Direction cosines of parallel vectors:

  Let $\vec{a}$ and $\vec{b}$ be two parallel vectors. Then $\vec{b} = \lambda\vec{a}$ for some λ.

  If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then $\vec{b} = \lambda\vec{a} \Rightarrow \vec{b} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

  This shows that $\vec{b}$ has direction ratios λa_1, λa_2 and λa_3, i.e., a_1, a_2 and a_3 because $\lambda a_1 : \lambda a_2 : \lambda a_3 = a_1 : a_2 : a_3$. Thus, $\vec{a}$ and $\vec{b}$ have equal direction ratios and hence equal direction cosines too.

- If the direction ratios of $\vec{r}$ are a, b and c, then $\vec{r} = \dfrac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i}+b\hat{j}+c\hat{k})$.
- Projections of $\vec{r}$ on the coordinate axes are: $l|\vec{r}|$, $m|\vec{r}|$ and $n|\vec{r}|$.
- The projection of a segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line with direction cosines l, m and n is $(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two concurrent lines, then the direction cosines of the lines bisecting the angles between them are proportional to $l_1 \pm l_2$, $m_1 \pm m_2$ and $n_1 \pm n_2$.
- Acute angle θ between the two lines having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 is given by

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|,\ \sin\theta = \sqrt{(l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2}$$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines, then the acute angle θ between them is given by $\cos\theta = \dfrac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$,

$$\sin\theta = \frac{\sqrt{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

- Two lines having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are
  a. perpendicular if and only if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.
  b. parallel if and only if $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$
- Two lines having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are
  a. perpendicular if and only if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$
  b. parallel if and only if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

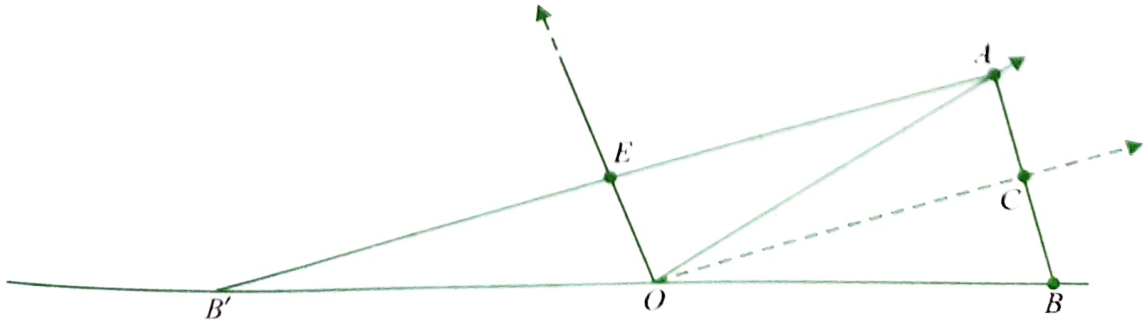
Direction Ratio of Line Along the Bisector of Two Given Lines

If l_1, m_1 and n_1 and l_2, m_2 and n_2 are the direction cosines of the two lines inclined to each other at an angle θ, then the direction cosines of the

a. internal bisector of the angle between these lines are $\dfrac{l_1+l_2}{2\cos(\theta/2)}$, $\dfrac{m_1+m_2}{2\cos(\theta/2)}$ and $\dfrac{n_1+n_2}{2\cos(\theta/2)}$, and

b. external bisector of the angle between these lines are $\dfrac{l_1-l_2}{2\sin(\theta/2)}$, $\dfrac{m_1-m_2}{2\sin(\theta/2)}$ and $\dfrac{n_1-n_2}{2\sin(\theta/2)}$.

Proof:



Let OA and OB be two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2. Let $OA = OB = 1$. Then the coordinates of A and B are (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively. Let OC be the bisector of $\angle AOB$. Then C is the midpoint of AB and so its coordinates are

$$\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$$

Therefore, the direction ratios of OC are $\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}$ and $\frac{n_1+n_2}{2}$. We have

$$OC = \sqrt{\left(\frac{l_1+l_2}{2}\right)^2 + \left(\frac{m_1+m_2}{2}\right)^2 + \left(\frac{n_1+n_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(l_1^2+m_1^2+n_1^2) + (l_2^2+m_2^2+n_2^2) + 2(l_1l_2+m_1m_2+n_1n_2)}$$

$$= \frac{1}{2}\sqrt{2+2\cos\theta} \qquad (\because \cos\theta = l_1l_2 + m_1m_2 + n_1n_2)$$

$$= \frac{1}{2}\sqrt{2(1+\cos\theta)} = \cos\left(\frac{\theta}{2}\right)$$

Therefore, the direction cosines of $\overrightarrow{OC}$ are $\frac{l_1+l_2}{2(OC)}, \frac{m_1+m_2}{2(OC)}, \frac{n_1+n_2}{2(OC)}$

or $$\frac{l_1+l_2}{2\cos(\theta/2)}, \frac{m_1+m_2}{2\cos(\theta/2)}, \frac{n_1+n_2}{2\cos(\theta/2)}$$

In the figure, OE is the external bisector.

The coordinates of E are $\frac{l_1-l_2}{2}, \frac{m_1-m_2}{2}$ and $\frac{n_1-n_2}{2}$.

Therefore, direction ratios of OE are $\frac{l_1-l_2}{2}, \frac{m_1-m_2}{2}$ and $\frac{n_1-n_2}{2}$.

Also, $$OE = \frac{1}{2}\sqrt{2-2\cos\theta}$$

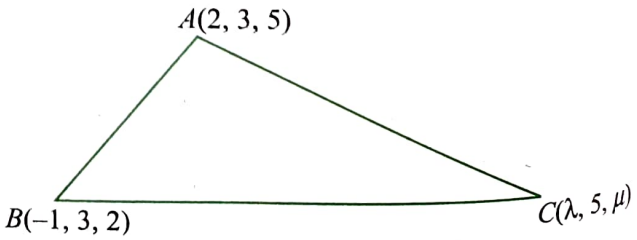
$$= \frac{1}{2}\sqrt{2(1-\cos\theta)}$$

$$= \sin(\theta/2)$$

Therefore, the direction cosines of $\overrightarrow{OE}$ are $\frac{l_1-l_2}{2\sin(\theta/2)}, \frac{m_1-m_2}{2\sin(\theta/2)}$ and $\frac{n_1-n_2}{2\sin(\theta/2)}$.

ILLUSTRATION 4.1

ABC is a triangle and $A = (2, 3, 5)$, $B = (-1, 3, 2)$ and $C = (\lambda, 5, \mu)$. If the median through A is equally inclined to the axes, then find the value of λ and μ.



Sol. Midpoint of BC is $\left(\frac{\lambda-1}{2}, 4, \frac{2+\mu}{2}\right)$

Direction ratios of the median through A are

$$\frac{\lambda-1}{2}-2, 4-3 \text{ and } \frac{2+\mu}{2}-5, \text{ i.e., } \frac{\lambda-5}{2}, 1 \text{ and } \frac{\mu-8}{2}.$$

The median is equally inclined to the axes; so the direction ratios must be equal. Therefore,

$$\frac{\lambda-5}{2}=1=\frac{\mu-8}{2} \Rightarrow \lambda=7, \mu=10$$

ILLUSTRATION 4.2

A line passes through the points $(6, -7, -1)$ and $(2, -3, 1)$. Find the direction cosines of the line if the line makes an acute angle with the positive direction of the x-axis.

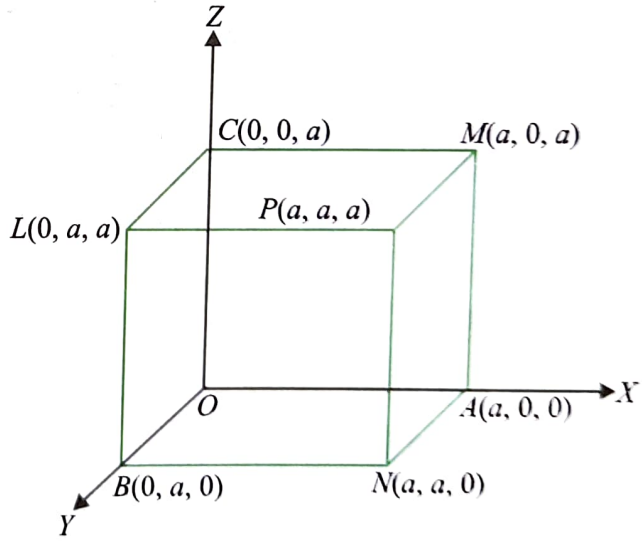
Sol. Let l, m and n be the direction cosines of the given line. As it makes an acute angle with the x-axis, $l>0$. The line passes through $(6, -7, -1)$ and $(2, -3, 1)$; therefore, its direction ratios are $(6-2, -7+3, -1-1)$ or $(4, -4, -2)$. Hence, the direction cosines of the given line are $2/3$, $-2/3$ and $-1/3$.

ILLUSTRATION 4.3

A line makes angles α, β, γ and δ with the diagonals of a cube. Show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=4/3.$$

Sol. The four diagonals of a cube are AL, BM, CN and OP.



Direction cosines of OP are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of AL are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of BM are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of CN are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{-1}{\sqrt{3}}$.

Let l, m and n be the direction cosines of a line which is inclined at angles α, β, γ and δ, respectively, to the four diagonals; then

$$\cos\alpha = l\cdot\frac{1}{\sqrt{3}}+m\cdot\frac{1}{\sqrt{3}}+n\cdot\frac{1}{\sqrt{3}}$$

$$=\frac{l+m+n}{\sqrt{3}}$$

Similarly, $$\cos\beta=\frac{-l+m+n}{\sqrt{3}}$$

$$\cos\gamma = \frac{l-m+n}{\sqrt{3}}$$

$$\cos\delta = \frac{l+m-n}{\sqrt{3}}$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = \frac{1}{3}[(l+m+n)^2+(-l+m+n)^2+(l-m+n)^2+(l+m-n)^2]$$

$$= \frac{1}{3}\cdot 4(l^2+m^2+n^2) = \frac{4}{3}$$

ILLUSTRATION 4.4

Find the angle between the lines whose direction cosines are given by $l+m+n=0$ and $2l^2+2m^2-n^2=0$.

Sol. $l^2+m^2+n^2=1$ (i)

$l+m+n=0$ (ii)

$2l^2+2m^2-n^2=0$ (iii)

$2(1-n^2)=n^2$ or $3n^2=2$ or $n=\pm\sqrt{2/3}$ (iv)

$2(l^2+m^2)=n^2=(-(l+m))^2$ or $l=m$ (v)

$l+m=\pm\sqrt{2/3}$ or $2l=\pm\sqrt{2/3}$

$l=\pm 1/\sqrt{6}$, $m=\pm 1/\sqrt{6}$

Direction cosines are

$$\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\sqrt{\frac{2}{3}}\right) \text{ and } \left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\sqrt{\frac{2}{3}}\right)$$

or

$$\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\sqrt{\frac{2}{3}}\right) \text{ and } \left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\sqrt{\frac{2}{3}}\right)$$

The angle between these lines in both the cases is $\cos^{-1}\left(-\frac{1}{3}\right)$.

ILLUSTRATION 4.5

A mirror and a source of light are situated at the origin O and at a point on OX, respectively. A ray of light from the source strikes the mirror and is reflected. If the direction ratios of the normal to the plane are 1, –1, 1, then find the DCs of the reflected ray.

Sol. Let the source of light be situated at $A(a, 0, 0)$, where $a \neq 0$.

Let OA be the incident ray and OB the reflected ray.

ON is the normal to the mirror at O. Therefore,

$$\angle AON = \angle NOB = \theta/2 \quad \text{(say)}$$

Direction ratios of OA are a, 0 and 0 and so its direction cosines are 1, 0 and 0.

Direction ratios of ON are $1/\sqrt{3}, -1/\sqrt{3}$ and $1/\sqrt{3}$. Therefore,

$$\angle AON = \angle NOB = (\theta/2) \quad \text{(say)}$$

$$\cos(\theta/2) = 1/\sqrt{3}$$

Let l, m and n be the direction cosines of the reflected ray OB. Then

$$\frac{l+1}{2\cos(\theta/2)} = \frac{1}{\sqrt{3}}, \frac{m+0}{2\cos(\theta/2)} = -\frac{1}{\sqrt{3}} \text{ and } \frac{n+0}{2\cos(\theta/2)} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad l = \frac{2}{3} - 1, m = \frac{-2}{3}, n = \frac{2}{3}$$

or $$l = -\frac{1}{3}, m = -\frac{2}{3}, n = \frac{2}{3}$$

CONCEPT APPLICATION EXERCISE 4.1

1. If the x-coordinate of a point P on the join of $Q(2, 2, 1)$ and $R(5, 1, -2)$ is 4, then find its z-coordinate.
2. If $\vec{r}$ is a vector of magnitude 21 and has direction ratios 2, –3 and 6, then find $\vec{r}$.
3. If $P(x, y, z)$ is a point on the line segment joining $Q(2, 2, 4)$ and $R(3, 5, 6)$ such that the projections of $\overrightarrow{OP}$ on the axes are 13/5, 19/5 and 26/5, respectively, then find the ratio in which P divides QR.
4. If O is the origin, $OP = 3$ with direction ratios –1, 2 and –2, then find the coordinates of P.
5. The line joining the points (–2, 1, –8) and (a, b, c) is parallel to the line whose direction ratios are 6, 2 and 3. Find the values of a, b and c.
6. Direction ratios of two lines are a, b, c and $1/bc$, $1/ca$, $1/ab$. Then the lines are _____.
7. Find the angle between the lines whose direction cosines are connected by the relations $l + m + n = 0$ and $2lm + 2nl - mn = 0$.

ANSWERS

1. –1
2. $6\hat{i} - 9\hat{j} + 18\hat{k}$
3. 3 : 2
4. –1, 2 and –2
5. $a = 6\lambda - 2, b = 2\lambda + 1, c = 3\lambda - 8$
6. parallel
7. $2\pi/3$

LINE IN SPACE

EQUATION OF STRAIGHT LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

Vector Form

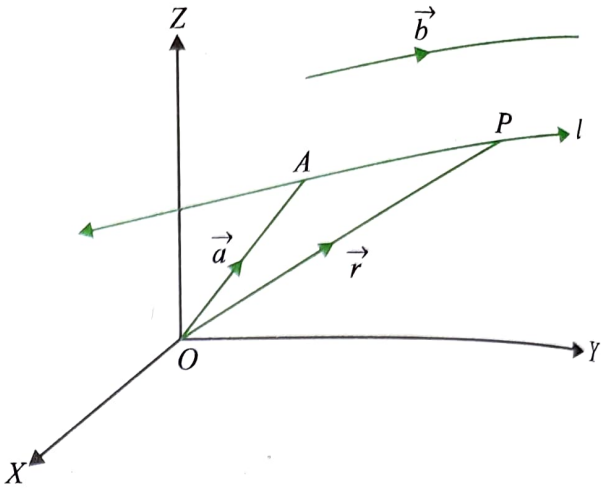
Line Passing through Point A($\vec{a}$) and Parallel to Vector $\vec{b}$

Let A be the given point and let AP be the given line through A.

Let $\vec{b}$ be any vector parallel to the given line.

Position vector of point A is $\vec{a}$.

Let P be any point on line AP, and let its position vector be $\vec{r}$.



Then, we have $\vec{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + \lambda\vec{b}$

(where $\overrightarrow{AP} = \lambda\vec{b}$).

Hence, the vector equation of straight line, $\vec{r} = \vec{a} + \lambda\vec{b}$. (i)

Here, $\vec{r}$ is the position vector of any point $P(x, y, z)$ on the line. So $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

In particular, the equation of straight line through origin and parallel to $\vec{b}$ is $\vec{r} = \lambda\vec{b}$.

Cartesian Form

Let the coordinates of the given point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b and c. Consider the coordinates of any point P be (x, y, z). Then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \text{ and } \vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$$

Substituting these values in (i) and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a;\ y = y_1 + \lambda b;\ z = z_1 + \lambda c$$

These are parametric equations of the line.

Eliminating the parameter λ, we get $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.

Note:

- Here any point on the line is $(x, y, z) \equiv (x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$ (λ being a parameter).
- Since the x-, y- and z-axes pass through the origin and have direction cosines $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, their equations are

Equation of x-axis : $\dfrac{x-0}{1} = \dfrac{y-0}{0} = \dfrac{z-0}{0}$ or $y = 0, z = 0$

Equation of y-axis : $\dfrac{x-0}{0} = \dfrac{y-0}{1} = \dfrac{z-0}{0}$ or $x = 0, z = 0$

Equation of z-axis : $\dfrac{x-0}{0} = \dfrac{y-0}{0} = \dfrac{z-0}{1}$ or $x = 0, y = 0$

EQUATION OF LINE PASSING THROUGH TWO GIVEN POINTS

Vector Form

From the figure, $\overrightarrow{OP} = \vec{r}$, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

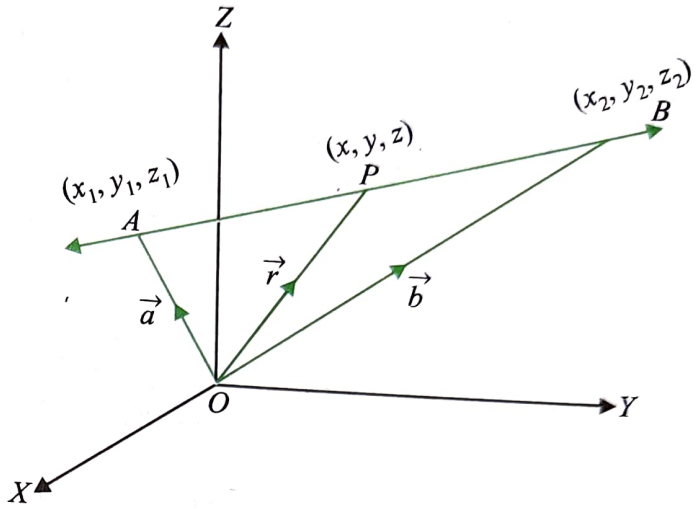
Since $\overrightarrow{AP}$ is collinear with $\overrightarrow{AB}$, $\overrightarrow{AP} = \lambda \overrightarrow{AB}$ for some scalar λ, we have

$$\overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$$

or $$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

or $$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \text{(i)}$$

Therefore, the equation of a straight line passing through $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.



Cartesian Form

We have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

Substituting these values in (i), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

Equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda(x_2 - x_1); y = y_1 + \lambda(y_2 - y_1); z = z_1 + \lambda(z_2 - z_1)$$

On eliminating λ, we obtain $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1} = \lambda$

which is the equation of the line in Cartesian form.

ILLUSTRATION 4.6

The Cartesian equation of a line is $\dfrac{x-3}{2} = \dfrac{y+1}{-2} = \dfrac{z-3}{5}$. Find the vector equation of the line.

Sol. The given line is $\dfrac{x-3}{2} = \dfrac{y+1}{-2} = \dfrac{z-3}{5}$.

Note that it passes through $(3, -1, 3)$ and is parallel to the line whose direction ratios are 2, –2 and 5. Therefore, its vector equation is $\vec{r} = 3\hat{i} - \hat{j} + 3\hat{k} + \lambda(2\hat{i} - 2\hat{j} + 5\hat{k})$, where λ is a parameter.

ILLUSTRATION 4.7

The Cartesian equations of a line are $6x - 2 = 3y + 1 = 2z - 2$. Find its direction ratios and also find a vector equation of the line.

Sol. The given line is $6x - 2 = 3y + 1 = 2z - 2$ (i)

To put it in the symmetrical form, we must make the coefficients of x, y and z as 1. To do this, we divide each of the expressions in (i) by 6 and obtain $\frac{x-(1/3)}{1} = \frac{y+(1/3)}{2} = \frac{z-1}{3}$.

This shows that the given line passes through $(1/3, -1/3, 1)$ and is parallel to the line whose direction ratios are 1, 2 and 3. Therefore, its vector equation is

$$\vec{r} = \frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$$

ILLUSTRATION 4.8

A line passes through the point with position vector $2\hat{i} - 3\hat{j} + 4\hat{k}$ and is in the direction of $3\hat{i} + 4\hat{j} - 5\hat{k}$. Find the equations of the line in vector and Cartesian forms.

Sol. Since the line passes through $2\hat{i} - 3\hat{j} + 4\hat{k}$ and has direction of $3\hat{i} + 4\hat{j} - 5\hat{k}$, its vector equation is

$$\vec{r} = \hat{a} + \lambda\hat{b} \Rightarrow \vec{r} = 2\hat{i} - 3\hat{j} + 4\hat{k} + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k}), \text{ where } \lambda \text{ is a parameter.} \quad \text{(i)}$$

Cartesian equivalent of (i) is $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-4}{-5}$

ILLUSTRATION 4.9

Find the vector equation of line passing through $A(3, 4, -7)$ and $B(1, -1, 6)$. Also find its cartesian equations.

Sol. Since the line passes through $A(3\hat{i} + 4\hat{j} - 7\hat{k})$ and $B(\hat{i} - \hat{j} + 6\hat{k})$, its vector equation is

$$\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + \lambda[(\hat{i} - \hat{j} + 6\hat{k}) - (3\hat{i} + 4\hat{j} - 7\hat{k})]$$

$$= 3\hat{i} + 4\hat{j} - 7\hat{k} - \lambda(2\hat{i} + 5\hat{j} - 13\hat{k}) \quad \text{(i)}$$

where λ is a parameter.

The Cartesian equivalent of (i) is $\frac{x-3}{2} = \frac{y-4}{5} = \frac{z+7}{-13}$.

ILLUSTRATION 4.10

Find Cartesian and vector equation of the line which passes through the point $(-2, 4, -5)$ and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

Sol. Equation of given line is $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ (i)

Direction ratios of line (i) are 3, 5, 6

Therefore, direction ratios of required line are also 3, 5, 6.

Required line passes through the point $(-2, 4, -5)$.

Hence, equation of required line is $\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$

Its vector form is $\vec{r}=(-2\hat{i}+4\hat{j}-5\hat{k})+\lambda(3\hat{i}+5\hat{j}+6\hat{k})$

ILLUSTRATION 4.11

Find the equation of a line which passes through the point (2, 3, 4) and which has equal intercepts on the axes.

Sol. Since line has equal intercepts on axes, it is equally inclined to axes.

$\Rightarrow$ Line is along the vector $a(\hat{i}+\hat{j}+\hat{k})$

$\Rightarrow$ Equation of line $=\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{1}$

ILLUSTRATION 4.12

Find the points where line $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{z}{1}$ intersects xy, yz and zx planes.

Sol. Line meets xy-plane where $z = 0$

Hence, from the given equation of line, $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{0}{1}$

$\Rightarrow$ $x = 1$ and $y = -2$.

$\Rightarrow$ Line meets xy-plane at $(1, -2, 0)$.

Line meets yz-plane where $x = 0$

Hence, from the given equation of line, $\frac{0-1}{2}=\frac{y+2}{-1}=\frac{z}{1}$

$\Rightarrow$ $z=\frac{-1}{2}$ and $y=-\frac{3}{2}$

$\Rightarrow$ Line meets yz-plane at $\left(0,-\frac{3}{2},\frac{-1}{2}\right)$

Line meets zx-plane where $y = 0$

Hence, from the given equation of line $\frac{x-1}{2}=\frac{0+2}{-1}=\frac{z}{1}$

$\Rightarrow$

$\Rightarrow$ $z = -2, x = -3$

Line meets zx-plane at $(-3, 0, -2)$

ILLUSTRATION 4.13

Find the equation of line $x + y - z - 3 = 0 = 2x + 3y + z + 4$ in symmetric form. Find the direction ratios of the line.

Sol. In the section of planes we will see that equation of the form $ax + by + cz + d = 0$ is the equation of the plane in the space.

Now equation of line in the form $x+y-z-3=0=2x+3y+z+4$ means set of those points in space which are common to the planes $x+y-z-3=0$ and $2x+3y+z+4=0$, which lie on the line of intersection of planes. For example, equation of x-axis is $y=z=0$ where xy-plane ($z=0$) and xz-plane ($y=0$) intersect.

Now to get the equation of line in symmetric form , in above equations, first we eliminate any one of the variables, say z. Then adding $x+y-z-3=0$ and $2x+3y+z+4=0$, we get

$$3x+4y+1=0 \text{ or } 3x=-4y-1=\lambda \text{ (say)}$$

or $$x=\frac{\lambda}{3}, y=\frac{\lambda+1}{-4}$$

Putting these values in $x+y-z-3=0$, we have $\frac{\lambda}{3}+\frac{\lambda+1}{-4}-z-3=0$

or $$\lambda=39+12z$$

Comparing values of λ, we have equation of line as

$$3x=-4y-1=12z+39$$

or $$\frac{3x}{12}=\frac{-4y-1}{12}=\frac{12z+39}{12} \quad \text{or} \quad \frac{x}{4}=\frac{y+\frac{1}{4}}{-3}=\frac{z+\frac{13}{4}}{1}$$

Hence, the line is passing through point $\left(0,-\frac{1}{4},-\frac{13}{4}\right)$ and have direction ratios $4, -3, 1$.

If we eliminate x or y first we will get equation of line having same direction ratio but with different point on the line.

ILLUSTRATION 4.14

Find the vector equation of line passing through the point $(1, 2, -4)$ and perpendicular to the two lines:

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

Sol. Since the line to be determined is perpendicular to the given two straight lines, it is directed towards

$$\text{Vector } (3\hat{i}-16\hat{j}+7\hat{k})\times(3\hat{i}+8\hat{j}-5\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = 24\hat{i}+36\hat{j}+72\hat{k}$$

Therefore, direction ratios of the line are 24, 36, 72 or 2, 3, 6.

Hence, equation of line passing through the point $(1, 2, -4)$ and parallel to the vector $2\hat{i}+3\hat{j}+6\hat{k}$ is

$$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$$

This is the equation of the required line.

ILLUSTRATION 4.15

If $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\mu(\hat{i}+\hat{j}-\hat{k})$ are two lines, then find the equation of acute angle bisector of two lines.

Sol. Lines are $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\mu(\hat{i}+\hat{j}-\hat{k})$ along vectors $(\hat{i}-\hat{j}+\hat{k})$ and $(\hat{i}+\hat{j}-\hat{k})$, respectively.

Angle between two lines $= \cos^{-1}\left(\dfrac{(1)\times(1)+(-1)(1)+(1)(-1)}{\sqrt{3}\,\sqrt{3}}\right) = \cos^{-1}\left(-\dfrac{1}{3}\right)$

which is an obtuse angle.

$\therefore$ Vector along acute angle bisector $= \lambda\left[\dfrac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}} - \dfrac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}\right] = \dfrac{2\lambda}{\sqrt{3}}(-\hat{j}+\hat{k})$

$\therefore$ Equation of acute angle bisector $= (\hat{i}+2\hat{j}+3\hat{k})+t(\hat{j}-\hat{k})$

ILLUSTRATION 4.16

Find the equation of the line drawn through point (1, 0, 2) to meet the line $\dfrac{x+1}{3}=\dfrac{y-2}{-2}=\dfrac{z+1}{-1}$ at right angles.

Sol. Given line is $\overleftrightarrow{AB} \equiv \dfrac{x+1}{3}=\dfrac{y-2}{-2}=\dfrac{z+1}{-1}$ (i)

Let $P \equiv (1, 0, 2)$

Any point on line (i) is $Q \equiv (3r-1, -2r+2, -r-1)$

Direction ratios of PQ are $3r-2, -2r+2, -r-3$

Direction ratios of line AB are $3, -2, -1$

$\because$ $PQ \perp AB$

$\therefore$ $3(3r-2)-2(-2r+2)-1(-r-3)=0$

or $14r = 7$ or $r = \dfrac{1}{2}$

Therefore, direction ratios of PQ are $-\dfrac{1}{2}, 1, -\dfrac{7}{2}$ or $1, -2, 7$

Equation of line $PQ \equiv \dfrac{x-1}{1}=\dfrac{y}{-2}=\dfrac{z-2}{7}$

ILLUSTRATION 4.17

Line L_1 is parallel to vector $\vec{\alpha}=-3\hat{i}+2\hat{j}+4\hat{k}$ and passes through a point $A(7, 6, 2)$ and line L_2 is parallel to a vector $\vec{\beta}=2\hat{i}+\hat{j}+3\hat{k}$ and passes through a point $B(5, 3, 4)$. Now a line L_3 parallel to a vector $\vec{r}=2\hat{i}-2\hat{j}-\hat{k}$ intersects the lines L_1 and L_2 at points C and D, respectively, then find $|\overrightarrow{CD}|$.

Sol. Line L_1 is parallel to vector $\vec{\alpha}=-3\hat{i}+2\hat{j}+4\hat{k}$ and passes through a point $A(7, 6, 2)$.

Therefore, position vector of any point on the line is $7\hat{i}+6\hat{j}+2\hat{k}+\lambda(-3\hat{i}+2\hat{j}+4\hat{k})$; line L_2 is parallel to a vector $\vec{\beta}=2\hat{i}+\hat{j}+3\hat{k}$ and passes through a point $B(5, 3, 4)$.

Position vector of any point on the line is $5\hat{i}+3\hat{j}+4\hat{k}+\mu(2\hat{i}+\hat{j}+3\hat{k})$

$\therefore$ $\overrightarrow{CD} = 2\hat{i}+3\hat{j}-2\hat{k}+\lambda(-3\hat{i}+2\hat{j}+4\hat{k})-\mu(2\hat{i}+\hat{j}+3\hat{k})$

Since it is parallel to $2\hat{i}-2\hat{j}-\hat{k}$, we have

$$\frac{2-3\lambda-2\mu}{2}=\frac{3+2\lambda-\mu}{-2}=\frac{-2+4\lambda-3\mu}{-1}$$

Solving these equations we get $\lambda = 2$ and $\mu = 1$. Therefore,

$$\overrightarrow{CD} = -6\hat{i} + 6\hat{j} + 3\hat{k}$$

or $$|\overrightarrow{CD}| = 9$$

ILLUSTRATION 4.18

Find the coordinates of a point on the line $\frac{x-1}{2} = \frac{y+1}{-3} = z$ at a distance $4\sqrt{14}$ from the point $(1, -1, 0)$.

Sol. Any point on the given line is $(2r + 1, -3r - 1, r)$ and its distance from $(1, -1, 0)$ is

$$(2r)^2 + (-3r)^2 + r^2 = \left(4\sqrt{14}\right)^2$$

or $$r = \pm 4$$

Thus, coordinates are $(9, -13, 4)$ and $(-7, 11, -4)$ and the point nearer to the origin is $(-7, 11, -4)$.

ANGLE BETWEEN TWO LINES

Let the given lines be

$$\left.\begin{aligned} \vec{r} &= \vec{a} + \lambda\vec{b} \quad \text{(i)} \\ \vec{r} &= \vec{a}' + \lambda\vec{b}' \quad \text{(ii)} \end{aligned}\right\} \rightarrow \text{Vector form}$$

$$\left.\begin{aligned} \frac{x-a_1}{b_1} &= \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} \\ \frac{x-a_1'}{b_1'} &= \frac{y-a_2'}{b_2'} = \frac{z-a_3'}{b_3'} \end{aligned}\right\} \rightarrow \text{Cartesian form}$$

Clearly (i) and (ii) are straight lines in the directions of $\vec{b}$ and $\vec{b}'$, respectively.
Let θ be the angle between the straight lines (i) and (ii).

Then θ is the angle between vectors $\vec{b}$ and $\vec{b}'$. Therefore,

$$\cos\theta = \frac{\vec{b}\cdot\vec{b}'}{|\vec{b}||\vec{b}'|}$$

$$\vec{b} = b\hat{i} + b_2\hat{j} + b_3\hat{k}, \quad \vec{b}' = b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}$$

$\therefore$ $$\vec{b}\cdot\vec{b}' = b_1b_1' + b_2b_2' + b_3b_3'$$

and $$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^3}, \quad |\vec{b}'| = \sqrt{b_1'^2 + b_2'^2 + b_3'^2}$$

$\Rightarrow$ $$\cos\theta = \frac{b_1b_1' + b_2b_2' + b_3b_3'}{\sqrt{b_1^2 + b_2^2 + b_3^3}\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}$$

Notes:

- If the lines are perpendicular, then $\vec{b}\cdot\vec{b}' = 0 \Rightarrow b_1b_1' + b_2b_2' + b_3b_3' = 0$.
- If the lines are parallel, then $\vec{b} = \lambda\vec{b}'$ for some scalar $\lambda \Rightarrow \frac{b_1}{b_2'} = \frac{b_2}{b_2'} = \frac{b_3}{b_3'}$.

ILLUSTRATION 4.19

Find the angle between the following pair of lines:

i. $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

ii. $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Sol.

i. The given lines are parallel to the vectors $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$, respectively. If θ is the angle between the given pair of lines, then

$$\cos\theta = \frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} = \frac{(3)(1)+(2)(2)+(6)(2)}{\sqrt{3^2+2^2+6^2}\sqrt{1^2+2^2+2^2}} = \frac{19}{7\times 3}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

ii. The given lines are parallel to the vectors $\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$, respectively. If θ is the angle between the given pair of lines, then

$$\cos\theta = \frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} = \frac{(2)(4)+(2)(1)+(1)(8)}{\sqrt{2^2+2^2+1^2}\sqrt{4^2+1^2+8^2}} = \frac{18}{3\times 9} = \frac{2}{3}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

ILLUSTRATION 4.20

Find the values of p so that line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

Sol. The given equations can be written in the standard form as

$$\frac{x-1}{-3} = \frac{y-2}{2p/7} = \frac{z-3}{2} \text{ and } \frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5}$$

The direction ratios of the lines are $-3, \frac{2p}{7}; 2$ and $\frac{-3p}{7}, 1, -5$, respectively.

Lines are perpendicular to each other. Therefore,

$$(-3)\cdot\left(\frac{-3p}{7}\right) + \left(\frac{2p}{7}\right)\cdot(1) + 2\cdot(-5) = 0$$

or

$$\frac{9p}{7} + \frac{2p}{7} = 10 \text{ or } 11p = 70 \text{ or } p = \frac{70}{11}$$

ILLUSTRATION 4.21

Find the acute angle between the lines $\frac{x-1}{l} = \frac{y+1}{m} = \frac{z}{n}$ and $\frac{x+1}{m} = \frac{y-3}{n} = \frac{z-1}{l}$, where $l > m > n$, and l, m, n are the roots of the cubic equation $x^3 + x^2 - 4x = 4$.

Sol. Given lines are $\frac{x-1}{l}=\frac{y+1}{m}=\frac{z}{n}$ and $\frac{x+1}{m}=\frac{y-3}{n}=\frac{z-1}{l}$

Angle between lines is given by

$$\cos\theta=\frac{lm+mn+nl}{l^2+m^2+n^2} \quad \text{(i)}$$

Now l, m, n are the roots of the cubic equation $x^3+x^2-4x=4$. Thus,

$$l+m+n=-1 \text{ and } lm+mn+nl=-4$$

$$(l+m+n)^2=l^2+m^2+n^2+2(lm+mn+nl)$$

or $$(-1)^2=l^2+m^2+n^2+2(-4)$$

or $$l^2+m^2+n^2=9$$

$\therefore$ $$\cos\theta=-\frac{4}{9}$$

Therefore, acute angle between the lines is $\cos^{-1}\frac{4}{9}$.

ILLUSTRATION 4.22

Find the condition if lines $x=ay+b, z=cy+d$ and $x=a'y+b', z=c'y+d'$ are perpendicular.

Sol. The equations of straight lines can be rewritten as

$$x=ay+b, z=cy+d \Rightarrow \frac{x-b}{a}=\frac{y-0}{1}=\frac{z-d}{c}$$

and $$x=a'y+b', z=c'y+d' \Rightarrow \frac{x-b'}{a'}=\frac{y-0}{1}=\frac{z-d'}{c'}$$

The above lines are perpendicular if $aa'+1\cdot1+c\cdot c'=0$.

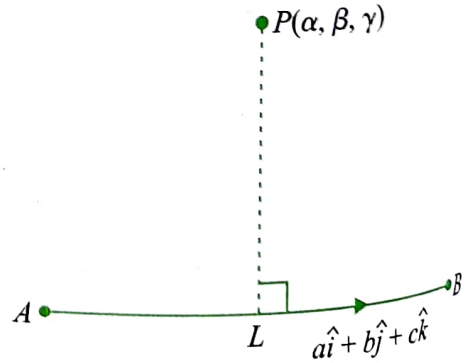
POINT AND LINE

Foot of Perpendicular from a Point on the Given Line

Cartesian Form

Here, the equation of line AB is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.

Let L be the foot of the perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$.



Let the coordiantes of L be $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$.

Then the direction ratios of PL are $(x_1+a\lambda-\alpha, y_1+b\lambda-\beta, z_1+c\lambda-\gamma)$.

Direction ratios of AB are (a, b, c).

Since PL is perpendicular to AB,

$$a(x_1+a\lambda-\alpha)+b(y_1+b\lambda-\beta)+c(z_1+c\lambda-\gamma)=0$$

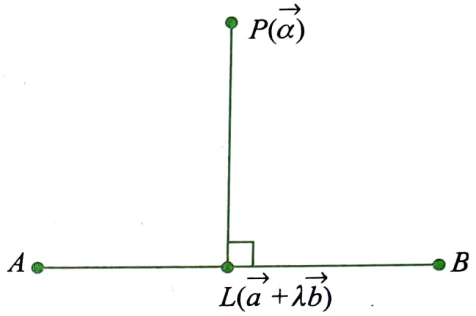
$$\lambda=\frac{a(\alpha-x_1)+b(\beta-y_1)+c(\gamma-z_1)}{a^2+b^2+c^2}$$

Putting the value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we get the foot of the perpendicular. Now we can get distance PL using the distance formula.

Vector Form

Let L be the foot of the perpendicular drawn from $P(\vec{\alpha})$ on the line $\vec{r} = \vec{a} + \lambda\vec{b}$.

Since $\vec{r}$ denotes the position vector of any point on the line $\vec{r} = \vec{a} + \lambda\vec{b}$, the position vector of L will be $(\vec{a} + \lambda\vec{b})$.



Directions ratios of $PL = \vec{a} - \vec{\alpha} + \lambda\vec{b}$

Since $\overrightarrow{PL}$ is perpendicular to $\vec{b}$, we have

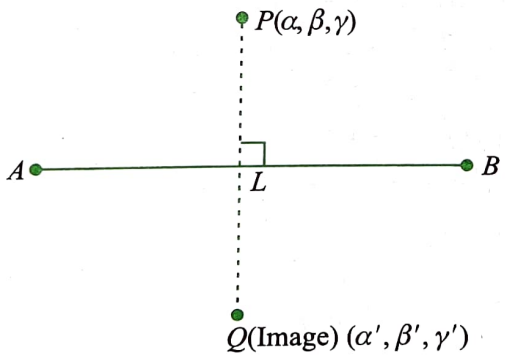
$$(\vec{a} - \vec{\alpha} + \lambda\vec{b}) \cdot \vec{b} = 0$$

or $$(\vec{a} - \vec{\alpha}) \cdot \vec{b} + \lambda\vec{b} \cdot \vec{b} = 0$$

or $$\lambda = \frac{-(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}$$

Thus, position vector of L is $\vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$, which is the foot of the perpendicular.

Image of a Point in the Given Line



Since L (foot of perpendicular) is the midpoint of P and Q (image of a point P in the line), we can get Q if L is found out.

ILLUSTRATION 4.23

Find the coordinates of the foot of the perpendicular drawn from point $A(1, 0, 3)$ to the join of points $B(4, 7, 1)$ and $C(3, 5, 3)$.

Sol. Let D be the foot of the perpendicular and let it divide BC in the ratio $\lambda : 1$. Then the coordinates of D are $\frac{3\lambda + 4}{\lambda + 1}, \frac{5\lambda + 7}{\lambda + 1}$ and $\frac{3\lambda + 1}{\lambda + 1}$.

Now, $\overrightarrow{AD} \perp \overrightarrow{BC} \Rightarrow \overrightarrow{AD} \cdot \overrightarrow{BC} = 0$

$\Rightarrow$ $(2\lambda + 3) + 2(5\lambda + 7) + 4 = 0$

or $\lambda = -\frac{7}{4}$

Thus, coordinates of D are $\frac{5}{3}, \frac{7}{3}$ and $\frac{17}{3}$.

ILLUSTRATION 4.24

Find the length of the perpendicular drawn from point (2, 3, 4) to line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.

Sol. Let P be the foot of the perpendicular from $A(2, 3, 4)$ to the given line l whose equation is

$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3} \quad \text{or} \quad \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = k \text{ (say)} \qquad \text{(i)}$$

$\therefore$ $x = 4 - 2k, y = 6k, z = 1 - 3k$

As P lies on (i), coordinates of P are $(4 - 2k, 6k, 1 - 3k)$ for some value of k.

The direction ratios of AP are

$$(4 - 2k - 2, 6k - 3, 1 - 3k - 4) \text{ or } (2 - 2k, 6k - 3, -3 - 3k)$$

Also, the direction ratios of l are -2, 6 and -3.

Since $AP \perp l$, we have

$$-2(2 - 2k) + 6(6k - 3) - 3(-3 - 3k) = 0$$

or $-4 + 4k + 36k - 18 + 9 + 9k = 0$ or $49k - 13 = 0$ or $k = 13/49$

We have $AP^2 = (4 - 2k - 2)^2 + (6k - 3)^2 + (1 - 3k - 4)^2$

$= (2 - 2k)^2 + (6k - 3)^2 + (-3 - 3k)^2$

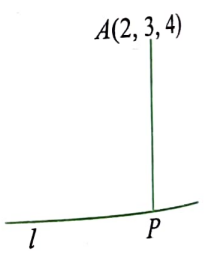
$= 4 - 8k + 4k^2 + 36k^2 - 36k + 9 + 9 + 18k + 9k^2$

$= 22 - 26k + 49k^2$

$= 22 - 26\left(\frac{13}{49}\right) + 49\left(\frac{13}{49}\right)^2$

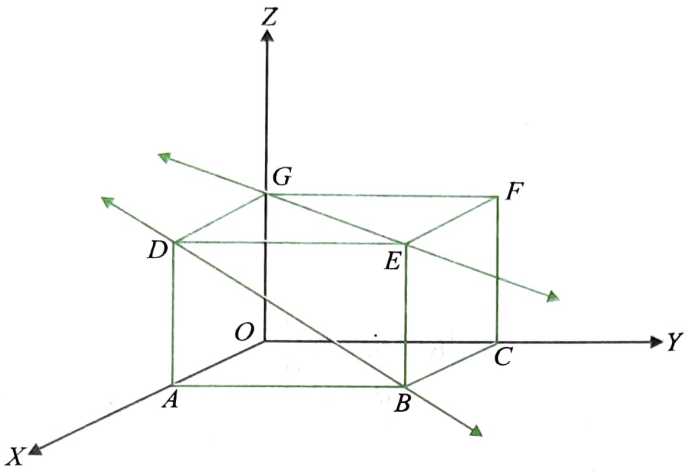
$= \frac{22 \times 49 - 26 \times 13 + 13^2}{49} = \frac{909}{49}$

$AP = \frac{3}{7}\sqrt{101}$



SHORTEST DISTANCE BETWEEN TWO LINES

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e., the length of the perpendicular drawn from any point on one line onto the other line. Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are *non-coplanar* and are called *skew lines*.



Line GE goes diagonally across the ceiling and line DB passes through one corner of the ceiling directly above A and goes diagonally down the wall. These lines are skew because they are not parallel and also never meet.

By the shortest distance between two lines, we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

Shortest Distance between Two Non-coplanar Lines

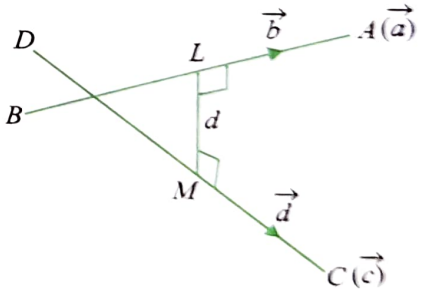
Vector Form

Let the given lines be $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r} = \vec{c} + t_1\vec{d}$.

If two lines AB and CD do not intersect, there is always a line intersecting both the lines perpendicularly.

The intercept on this line made by AB and CD is called the shortest distance between lines AB and CD.

In the figure, the shortest distance $= LM$, where $\angle ALM = \angle CML = 90°$.



In the figure, the shortest distance

$$LM = |\text{Projection of } \overrightarrow{AC} \text{ along } \overrightarrow{ML}|$$

$$= \left| \overrightarrow{AC} \cdot \frac{\overrightarrow{ML}}{|\overrightarrow{ML}|} \right| = \frac{|(\vec{a} - \vec{c}) \cdot \overrightarrow{LM}|}{|\overrightarrow{LM}|}$$

Now $\overrightarrow{LM}$ is perpendicular to both $\vec{b}$ and $\vec{d}$. Thus,

$$\overrightarrow{LM} = \vec{b} \times \vec{d}$$

$$= \frac{|(\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|}$$

$$= \frac{|[\vec{b}\ \vec{d}\ \vec{a} - \vec{c}]|}{|\vec{b} \times \vec{d}|}$$

Cartesian Form

Let the two skew lines be $\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}$ and $\frac{x-c_1}{d_1}=\frac{y-c_2}{d_2}=\frac{z-c_3}{d_3}$

Then $$\text{Shortest distance} = \frac{\left\|\begin{vmatrix} c_1-a_1 & c_2-a_2 & c_3-a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\right\|}{\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\right\|}$$

Condition for Lines to Intersect

Two lines $\vec{r}=\vec{a}+t\vec{b}$ and $\vec{r}=\vec{c}+t_1\vec{d}$ are intersecting if

$$\left|\frac{(\vec{a}-\vec{c})\cdot(\vec{b}-\vec{d})}{\vec{b}\times\vec{d}}\right|=0$$

$$\Rightarrow \quad \begin{vmatrix} c_1-a_1 & c_2-a_2 & c_3-a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}=0$$

Distance between Two Parallel Lines

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be given by

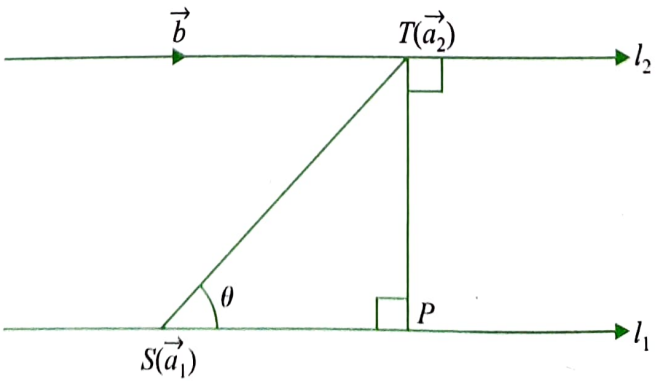
$$\vec{r}=\vec{a_1}+\lambda\vec{b} \quad \text{(i)}$$

$$\vec{r}=\vec{a_2}+\mu\vec{b} \quad \text{(ii)}$$

where $\vec{a_1}$ is the position vector of a point S on l_1 and $\vec{a_2}$ is the position vector of a point T on l_2.

As l_1 and l_2 are coplanar, if the foot of the perpendicular from T on line l_1 is P, then the distance between the lines l_1 and $l_2 = |TP|$.

Let θ be the angle between vectors $\overrightarrow{ST}$ and $\vec{b}$.



$$\vec{b} \times \overrightarrow{ST} = (|\vec{b}||\overrightarrow{ST}|\sin\theta)\hat{n} \qquad \text{(iii)}$$

Then where $\hat{n}$ is the unit vector perpendicular to the plane of the lines l_1 and l_2.

But $$\overrightarrow{ST} = \vec{a}_2 - \vec{a}_1$$

Therefore, from (iii), we get

$$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|PT\,\hat{n} \qquad (\text{since } PT = ST\sin\theta)$$

i.e., $$|\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|PT \cdot 1 \qquad (\text{as } |\hat{n}| = 1)$$

Hence, the distance between the given parallel lines is

$$d = |\overrightarrow{PT}| = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

ILLUSTRATION 4.25

Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Also obtain the equation of the line of the shortest distance.

Sol. (i) The two given lines are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r_1$ (say) (i)

and $$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} = r_2 \text{ (say)} \qquad \text{(ii)}$$

Any point on (i) is given by $P\,(2r_1 + 1, 3r_1 + 2, 4r_1 + 3)$

And any point on (ii) is given by $Q\,(3r_2 + 2, 4r_2 + 4, 5r_2 + 5)$

Direction ratios of PQ are given by $3r_2 - 2r_1 + 1$, $4r_2 - 3r_1 + 2$ and $5r_2 - 4r_1 + 2$

Since PQ is perpendicular to (i), we get

or $$2\,(3r_2 - 2r_1 + 1) + 3(4r_2 - 3r_1 + 2) + 4\,(5r_2 - 4r_1 + 2) = 0$$

$$38r_2 - 29r_1 + 16 = 0 \qquad \text{(iii)}$$

Also PQ is perpendicular to (ii), we get

or $$3(3r_2 - 2r_1 + 1) + 4\,(4r_2 - 3r_1 + 2) + 5(5r_2 - 4r_1 + 2) = 0$$

$$50r_2 - 38r_1 + 21 = 0 \qquad \text{(iv)}$$

Solving (iii) and (iv), we obtain $r_2 = -(1/6)$, $r_1 = (1/3)$.

Therefore, coordinates of P and Q are $\left(\frac{5}{3}, 3, \frac{13}{3}\right)$ and $\left(\frac{3}{2}, \frac{10}{3}, \frac{25}{6}\right)$, respectively. Thus,

or $$PQ^2 = \left(\frac{3}{2} - \frac{5}{3}\right)^2 + \left(\frac{10}{3} - 3\right)^2 + \left(\frac{25}{6} - \frac{13}{3}\right)^2 = \left(-\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{6}\right)^2 = \frac{1}{6}$$

$$PQ = \frac{1}{\sqrt{6}}$$

The equation of the line of the shortest distance is given by

$$\frac{x-(5/3)}{(3/2)-(5/3)}=\frac{y-3}{(10/3)-3}=\frac{z-(13/3)}{(25/6)-(13/3)}$$

$$\frac{x-(5/3)}{-(1/6)}=\frac{y-3}{(1/3)}=\frac{z-(13/3)}{-(1/6)}$$

$$\frac{x-(5/3)}{1}=\frac{y-3}{-2}=\frac{z-(13/3)}{1}$$

Alternative method for finding the shortest distance:

Line (i) is passing through the point $(x_1, y_1, z_1) \equiv (1, 2, 3)$ and is parallel to vector

$$a_1\hat{i}+b_1\hat{j}+c_1\hat{k} \equiv 2\hat{i}+3\hat{j}+4\hat{k}$$

Line (ii) is passing through the point $(x_2, y_2, z_2) \equiv (2, 4, 5)$ and is parallel to the vector

$$a_2\hat{i}+b_2\hat{j}+c_2\hat{k} \equiv 3\hat{i}+4\hat{j}+5\hat{k}$$

Hence, the shortest distance between the lines using the formula

$$\frac{\left|\begin{matrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right|}{\left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right|} \text{ is } \frac{\left|\begin{matrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{matrix}\right|}{\left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{matrix}\right|}=\frac{1}{\sqrt{6}}$$

ILLUSTRATION 4.26

Determine whether the following pair of lines intersect or not.

i. $\vec{r}=\hat{i}-\hat{j}+\lambda(2\hat{i}+\hat{k});\ \vec{r}=2\hat{i}-\hat{j}+\mu(\hat{i}+\hat{j}-\hat{k})$

ii. $\vec{r}=\hat{i}+\hat{j}-\hat{k}+\lambda(3\hat{i}-\hat{j});\ \vec{r}=4\hat{i}-\hat{k}+\mu(2\hat{i}+3\hat{k})$

Sol.

i. Here $\vec{a_1}=\hat{i}-\hat{j}$, $\vec{a_2}=2\hat{i}-\hat{j}$, $\vec{b_1}=2\hat{i}+\hat{k}$ and $\vec{b_2}=\hat{i}+\hat{j}-\hat{k}$

Now $$[\vec{a_2}-\vec{a_1}\ \vec{b_1}\ \vec{b_2}]=\begin{vmatrix} 2-1 & -1+1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=-1\neq 0$$

Thus, the two given lines do not intersect.

ii. Here $\vec{a_1} = \hat{i} + \hat{j} - \hat{k}$, $\vec{a_2} = 4\hat{i} - \hat{k}$, $\vec{b_1} = 3\hat{i} - \hat{j}$ and $\vec{b_2} = 2\hat{i} + 3\hat{k}$

$$\Rightarrow \quad [\vec{a_2} - \vec{a_1}\ \vec{b_1}\ \vec{b_2}] = \begin{vmatrix} 4-1 & 0-1 & -1+1 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0$$

Thus, the two given lines intersect. Let us obtain the point of intersection of the two given lines.

For some values of λ and μ, the two values of $\vec{r}$ must coincide. Thus,

$$\hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j}) = 4\hat{i} - \hat{k} + \mu(2\hat{i} + 3\hat{k})$$

or $$(3 + 2\mu - 3\lambda)\hat{i} + (\lambda - 1)\hat{j} + 3\mu\hat{k} = 0$$

or $$3 + 2\mu - 3\lambda = 0,\ \lambda - 1 = 0,\ 3\mu = 0$$

Solving, we obtain $\lambda = 1$ and $\mu = 0$

Therefore, the point of intersection is $\vec{r} = 4\hat{i} - \hat{k}$ (by putting $\mu = 0$ in the second equation).

ILLUSTRATION 4.27

Find the shortest distance between lines $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

Sol. Here lines (i) ane (ii) are passing through the points $\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$, respectively, and are parallel to the vector $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$.

Hence, the distance between the lines using the formula is given by

$$\frac{|\vec{b} \times (\vec{a_2} - \vec{a_1})|}{|\vec{b}|} = \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 1 & -3 & -2 \end{vmatrix}}{3} = \frac{|4\hat{i} - 6\hat{j} - 7\hat{k}|}{3} = \frac{\sqrt{16 + 36 + 49}}{3} = \frac{\sqrt{101}}{3}$$

ILLUSTRATION 4.28

If the straight lines $x = -1 + s, y = 3 - \lambda s, z = 1 + \lambda s$ and $x = \frac{t}{2}, y = 1 + t, z = 2 - t$, with parameters s and t, respectively, are coplanar, then find λ.

Sol. The given lines $\frac{x+1}{1} = \frac{y-3}{-\lambda} = \frac{z-1}{\lambda} = s$

and $$\frac{x-0}{1/2} = \frac{y-1}{1} = \frac{y-2}{-1} = t$$

are coplanar if $\begin{vmatrix} 0+1 & 1-3 & 2-1 \\ 1 & -\lambda & \lambda \\ 1/2 & 1 & -1 \end{vmatrix} = 0$ or $\begin{vmatrix} 1 & -2 & 1 \\ 1 & -\lambda & \lambda \\ 1/2 & 1 & -1 \end{vmatrix} = 0$

or $1(\lambda - \lambda) + 2\left(-1 - \frac{\lambda}{2}\right) + 1\left(1 + \frac{\lambda}{2}\right) = 0$

or $\lambda = -2$

ILLUSTRATION 4.29

Find the equation of a line which passes through the point (1, 1, 1) and intersects the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4}$.

Sol. Any line passing through the point (1, 1, 1) is $\frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c}$ (i)

This line intersects the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.

If $a : b : c \neq 2 : 3 : 4$ and $\begin{vmatrix} 1-1 & 2-1 & 3-1 \\ a & b & c \\ 2 & 3 & 4 \end{vmatrix} = 0$

$\Rightarrow$ $a - 2b + c = 0$ (ii)

Again, line (i) intersects line $\frac{x-(-2)}{1} = \frac{y-3}{2} = \frac{z-(-1)}{4}$.

If $a : b : c \neq 1 : 2 : 4$ and $\begin{vmatrix} -2-1 & 3-1 & -1-1 \\ a & b & c \\ 1 & 2 & 4 \end{vmatrix} = 0$

$\Rightarrow$ $6a + 5b - 4c = 0$ (iii)

From (ii) and (iii) by cross multiplication, we have $\frac{a}{8-5} = \frac{b}{6+4} = \frac{c}{5+12}$

or $\frac{a}{3} = \frac{b}{10} = \frac{c}{17}$

So, the required line is $\frac{x-1}{3} = \frac{y-1}{10} = \frac{z-1}{17}$

CONCEPT APPLICATION EXERCISE 4.2

1. Find the point where line which passes through point (1, 2, 3) and is parallel to line $\vec{r} = \hat{i} - \hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$ meets the xy-plane.
2. Find the equation of the line passing through the points (1, 2, 3) and (−1, 0, 4).
3. Find the equation of the line passing through the point (2, −1, −1) and parallel to the line $-6x - 2 = 3y + 1 = 2z - 2$.

4. Find the equation of the line passing through the point $(-1, 2, 3)$ and perpendicular to the lines $\frac{x}{2}=\frac{y-1}{-3}=\frac{z+2}{-2}$ and $\frac{x+3}{-1}=\frac{y+3}{2}=\frac{z-1}{3}$.

5. Find the equation of the line passing through the intersection of $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ and also through the point $(2, 1, -2)$.

6. The straight line $\frac{x-3}{3}=\frac{y-2}{1}=\frac{z-1}{0}$ is

   a. parallel to the x-axis
   b. parallel to the y-axis
   c. parallel to the z-axis
   d. perpendicular to the z-axis

7. Find the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$.

8. If the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$ are at right angle, then find the value of k.

9. The equations of motion of a rocket are $x = 2t$, $y = -4t$ and $z = 4t$, where time t is given in seconds, and the coordinates of a moving point in kilometres. What is the path of the rocket? At what distance will be the rocket from the starting point $O(0, 0, 0)$ in 10 s?

10. Find the length of the perpendicular drawn from the point $(5, 4, -1)$ to the line $\vec{r}=\hat{i}+\lambda(2\hat{i}+9\hat{j}+5\hat{k})$, where λ is a parameter.

11. Find the image of point $(1, 2, 3)$ in the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$.

12. Find the shortest distance between the lines $\vec{r}=(1-\lambda)\hat{i}+(\lambda-2)\hat{j}+(3-2\lambda)\hat{k}$ and $\vec{r}=(\mu+1)\hat{i}+(2\mu-1)\hat{j}-(2\mu+1)\hat{k}$.

13. Find the shortest distance between the z-axis and the line, $x + y + 2z - 3 = 0$, $2x + 3y + 4z - 4 = 0$.

14. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then find the value of k.

15. Let O_1 and O_2 be the two skew lines. If P, Q are two distinct points on O_1 and R, S are two distinct points on O_2, then prove that PR cannot be parallel to QS.

ANSWERS

1. $(0, 4, 0)$
2. $\frac{x+1}{-2}=\frac{y-0}{-2}=\frac{z-4}{1}$
3. $\frac{x-2}{-1}=\frac{y+1}{2}=\frac{z+1}{3}$
4. $\frac{x+1}{5}=\frac{y-2}{4}=\frac{z-3}{-1}$
5. $\frac{x+1}{3}=\frac{y+1}{2}=\frac{z+1}{-1}$
6. (d)
7. 90°
8. $k=-\frac{10}{7}$
9. Path of the rocket is a straight line passing through origin; 60 km
10. $\sqrt{\frac{2109}{110}}$
11. $(5, 8, 15)$
12. $\frac{1}{\sqrt{5}}$
13. 2
14. $k = 9/2$

PLANE IN SPACE

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface.

A plane is determined uniquely if:

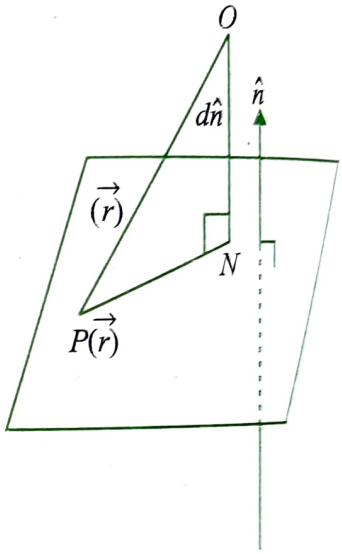
i. The normal to the plane and its distance from the origin is given, i.e., the equation of a plane in normal form.

ii. It passes through a point and is perpendicular to a given direction.

iii. It passes through three given non-collinear points.

EQUATION OF A PLANE IN NORMAL FORM

Consider a plane whose perpendicular distance from the origin is d $(d \neq 0)$.



If $\overrightarrow{ON}$ is the normal from the origin to the plane, and $\hat{n}$ is the unit normal vector along $\overrightarrow{ON}$, then $\overrightarrow{ON} = d\hat{n}$. Let P be any point on the plane. Therefore, $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$.

$$\therefore \quad \overrightarrow{NP} \cdot \overrightarrow{ON} = 0 \qquad \text{(i)}$$

Let $\vec{r}$ be the position vector of the point P. Then

$$\overrightarrow{NP} = \vec{r} - d\hat{n} \quad (\text{as } \overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OP})$$

Therefore, (i) becomes

$$(\vec{r} - d\hat{n}) \cdot d\hat{n} = 0$$

or $$(\vec{r} - d\hat{n}) \cdot \hat{n} = 0 \quad (d \neq 0)$$

or $$(\vec{r} \cdot \hat{n}) - d\hat{n} \cdot \hat{n} = 0$$

or $$\vec{r} \cdot \hat{n} = d \quad (\text{as } \hat{n} \cdot \hat{n} = 1) \qquad \text{(ii)}$$

This is the vector form of the equation of the plane.

Cartesian form

Equation (ii) gives the vector equation of a plane, where $\hat{n}$ is the unit vector normal to the plane. Let $P(x, y, z)$ be any point on the plane. Then

$$\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Let l, m and n be the direction cosines of $\hat{n}$. Then

$$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$$

Therefore, (ii) gives

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = d \qquad \text{(iii)}$$

or $$lx + my + nz = d$$

This is the Cartesian equation of the plane in the normal form.

Note: Equation (iii) shows that if $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the Cartesian equation of the plane, where a, b and c are the direction ratios of the normal to the plane.

ILLUSTRATION 4.30

Find the equation of plane which is at a distance $\frac{4}{\sqrt{14}}$ from the origin and is normal to vector $2\hat{i} + \hat{j} - 3\hat{k}$.

Sol. Here $\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$. Then $\frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} - 3\hat{k}}{\sqrt{2^2 + 1^2 + (-3)^2}} = \frac{2\hat{i} + \hat{j} - 3\hat{k}}{\sqrt{14}}$

Hence, required equation of plane is $\vec{r} \cdot \frac{1}{\sqrt{14}}(2\hat{i} + \hat{j} - 3\hat{k}) = \pm\frac{1}{\sqrt{14}}$

or $\vec{r} \cdot (2\hat{i} + \hat{j} - 3\hat{k}) = \pm 1$ (vector form)

or $2x + y - 3z = \pm 1$ (Cartesian form)

ILLUSTRATION 4.31

Find the unit vector perpendicular to the plane $\vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 5$.

Sol. Vector normal to the plane is $\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$

Hence, unit vector perpendicular to the plane is $\frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$

ILLUSTRATION 4.32

Find the distance of the plane $2x - y - 2z - 9 = 0$ from the origin.

Sol. The plane can be put in vector form as $\vec{r} \cdot (2\hat{i} - \hat{j} - 2\hat{k}) = 9$, where $\vec{r} = 2\hat{i} - \hat{j} - 2\hat{k}$. Here

$$\vec{n} = 2\hat{i} - \hat{j} - 2\hat{k}$$

$$\Rightarrow \quad \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$$

Dividing equation throughout by 3, we have equation of plane in the normal form as

$$\vec{r} \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3} = 3,$$

in which 3 is the distance of the plane from the origin.

ILLUSTRATION 4.33

Find the vector equation of a line passing through $3\hat{i} - 5\hat{j} + 7\hat{k}$ and perpendicular to the plane $3x - 4y + 5z = 8$.

Sol. The given plane $3x - 4y + 5z = 8$ or $(3\hat{i} - 4\hat{j} + 5\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 8$.

This shows that $\vec{d} = 3\hat{i} - 4\hat{j} + 5\hat{k}$ is normal to the given plane.

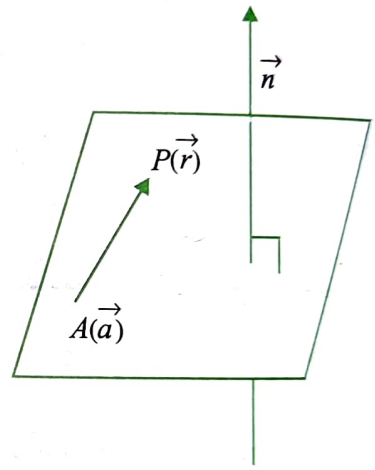
Therefore, the required line is parallel to $3\hat{i} - 4\hat{j} + 5\hat{k}$.

Since the required line passes through $3\hat{i} - 5\hat{j} + 7\hat{k}$, its equation is given by

$$\vec{r} = 3\hat{i} - 5\hat{j} + 7\hat{k} + \lambda(3\hat{i} - 4\hat{j} + 5\hat{k}), \text{ where } \lambda \text{ is a parameter.}$$

EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND NORMAL TO A GIVEN VECTOR

Vector Form



Suppose the plane passes through a point having position vector $\vec{a}$ and is normal to vector $\vec{n}$.

Then for any position of point $P(\vec{r})$ on the plane, $\overrightarrow{AP} \perp \vec{n}$. Thus,

$$\overrightarrow{AP} \cdot \vec{n} = 0$$

or $$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad (\because \overrightarrow{AP} = \vec{r} - \vec{a})$$

Hence, the required equation of the plane is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

Note:

The above equation can be written as $\vec{r} \cdot \vec{n} = d$, where $d = \vec{a} \cdot \vec{n}$ (known as scalar product form of plane).

The equation $\vec{r} \cdot \vec{n} = d$ is in normal form if $\vec{n}$ is a unit vector and d is the distance of the plane from the origin. If $\vec{n}$ is not a unit vector, then we reduce the equation $\vec{r} \cdot \vec{n} = d$ to the normal form by dividing both sides by $|\vec{n}|$; we get $\frac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = \frac{d}{|\vec{n}|}$ or $\vec{r} \cdot \hat{n} = \frac{d}{|n|} = p$ (distance from the origin).

Cartesian Form

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$, then

$$(\vec{r} - \vec{a}) = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$

Then equation of the plane can be written as

$$((x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = 0$$

or $$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Thus, the coefficients of x, y and z in the Cartesian equation of a plane are the direction ratios of the normal to the plane.

ILLUSTRATION 4.34

Find the equation of the plane passing through the point (2, 3, 1) having (5, 3, 2) as the direction ratios of the normal to the plane.

Sol. The equation of the plane passing through (x_1, y_1, z_1) and perpendicular to the line with direction ratios a, b and c is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Now, since the plane passes through (2, 3, 1) and is perpendicular to the line having direction ratios (5, 3, 2), the equation of the plane is given by $5(x - 2) + 3(y - 3) + 2(z - 1) = 0$ or $5x + 3y + 2z = 21$.

ILLUSTRATION 4.35

The foot of the perpendicular drawn from the origin to a plane is (1, 2, –3). Find the equation of the plane.

or

If O is the origin and the coordinates of P is (1, 2, –3), then find the equation of the plane passing through P and perpendicular to OP.

Sol. Since P (1, 2, –3) is the foot of the perpendicular from the origin to the plane, OP is normal to the plane. Thus, the direction ratios of normal to the plane are 1, 2 and –3.

Now, since the plane passes through (1, 2, –3), its equation is given by

$$1(x - 1) + 2(y - 2) - 3(z + 3) = 0$$

or $$x + 2y - 3z - 14 = 0$$

ILLUSTRATION 4.36

Find the equation of the plane such that image of point (1, 2, 3) in it is (–1, 0, 1) .

Sol. Since the image of A (1, 2, 3) in the plane is B (–1, 0, 1), the plane passes through the midpoint (0, 1, 2) of AB and is normal to the vector $\overrightarrow{AB} \equiv -2\hat{i} - 2\hat{j} - 2\hat{k}$.

Hence, the equation of the plane is $-2(x - 0) - 2(y - 1) - 2(z - 2) = 0$ or $x + y + z = 3$

EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

Cartesian form

Let the plane be passing through points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Let $P(x, y, z)$ be any point on the plane.

Then vectors $\overrightarrow{PA}$, $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are coplanar.

$$[\overrightarrow{PA}\ \overrightarrow{BA}\ \overrightarrow{CA}] = 0$$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0, \text{ which is the required equation of the plane}$$

Vector form

Vector form of the equation of the plane passing through three points A, B and C having position vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively.

Let $\vec{r}$ be the position vector of any point P in the plane.

Hence, vectors $\overrightarrow{AP} = \vec{r} - \vec{a}$ $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$ are coplanar, i.e.,

$$(\vec{r} - \vec{a}) \cdot \{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\} = 0$$

or $$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}) = 0$$

or $$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}) = 0$$

or $$\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}) = \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot (\vec{c} \times \vec{a})$$

or $$[\vec{r}\ \vec{b}\ \vec{c}] + [\vec{r}\ \vec{a}\ \vec{b}] + [\vec{r}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$$

which is the required equation of the plane.

Note:

- If p is the length of perpendicular from the origin on this plane, then $p = [\vec{a}\ \vec{b}\ \vec{c}]/n$, where $n = |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$.
- Four points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if $\vec{d}$ lies on the plane containing $\vec{a}, \vec{b}$ and $\vec{c}$, i.e.,

$$\vec{d} \cdot [\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$$

or $$[\vec{d}\ \vec{a}\ \vec{b}] + [\vec{d}\ \vec{b}\ \vec{c}] + [\vec{d}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$$

ILLUSTRATION 4.37

Find the equation of the plane passing through A (2, 2, –1), B (3, 4, 2) and C (7, 0, 6). Also find a unit vector perpendicular to this plane.

Sol. Here $(x_1, y_1, z_1) \equiv (2, 2, -1)$, $(x_2, y_2, z_2) \equiv (3, 4, 2)$ and $(x_3, y_3, z_3) \equiv (7, 0, 6)$

Then the equation of the plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x - 2 & y - 2 & z - (-1) \\ 3 - 2 & 4 - 2 & 2 - (-1) \\ 7 - 2 & 0 - 2 & 6 - (-1) \end{vmatrix} = 0$$

or $$5x + 2y - 3z = 17$$

A normal vector to this plane is $\vec{d} = 5\hat{i} + 2\hat{j} - 3\hat{k}$ (i)

Therefore, a unit vector normal to (i) is given by

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{5\hat{i} + 2\hat{j} - 3\hat{k}}{\sqrt{25 + 4 + 9}} = \frac{1}{\sqrt{38}}(5\hat{i} + 2\hat{j} - 3\hat{k})$$

ILLUSTRATION 4.38

Show that the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=0$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}+\hat{k})=0$ is equally inclined to $\hat{i}$ and $\hat{k}$. Also find the angle it makes with $\hat{j}$.

Sol. The line of intersection of the two planes will be perpendicular to the normals to the planes. Hence, it is parallel to the vector $(\hat{i}+2\hat{j}+3\hat{k})\times(3\hat{i}+2\hat{j}+\hat{k})=(-4\hat{i}+8\hat{j}-4\hat{k})$. Now,

$$(-4\hat{i}+8\hat{j}-4\hat{k})\cdot\hat{i}=-4 \text{ and } (-4\hat{i}+8\hat{j}-4\hat{k})\cdot\hat{k}=-4$$

Hence, the line is equally inclined to $\hat{i}$ and $\hat{k}$. Also,

$$\frac{(-4\hat{i}+8\hat{j}-4\hat{k})}{\sqrt{16+64+16}}\cdot\hat{j}=\frac{8}{\sqrt{96}}=\sqrt{\frac{2}{3}}$$

If θ is the required angle, then $\cos\theta=\sqrt{\frac{2}{3}}$ or $\theta=\cos^{-1}\sqrt{\frac{2}{3}}$.

EQUATION OF THE PLANE THAT PASSES THROUGH POINT A WITH POSITION VECTOR $\vec{a}$ AND IS PARALLEL TO GIVEN VECTORS $\vec{b}$ AND $\vec{c}$

Vector Form

Let $\vec{r}$ be the position vector of any point P in the plane. Then

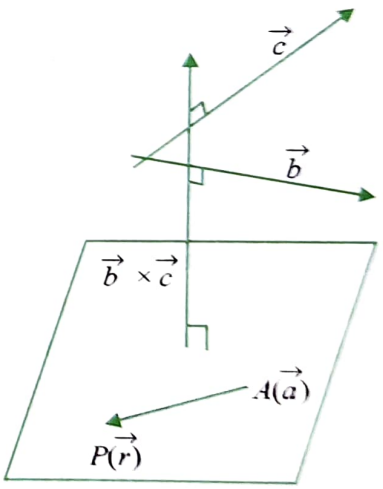
$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\vec{r}-\vec{a}$$

Since vectors $\vec{r}-\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, we have

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$$

or
$$\vec{r}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot(\vec{b}\times\vec{c}) \quad \text{or} \quad [\vec{r}\ \vec{b}\ \vec{c}]=[\vec{a}\ \vec{b}\ \vec{c}]$$

which is the required equation of the plane.



Cartesian Form

From $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$, we have $[\vec{r}-\vec{a}\ \vec{b}\ \vec{c}]$

$$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0$$, which is the required equation of the plane,

where $\vec{b}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$ and $\vec{c}=x_3\hat{i}+y_3\hat{j}+z_3\hat{k}$.

ILLUSTRATION 4.39

Find the vector equation of the following planes in cartesian form:

$$\vec{r}=\hat{i}-\hat{j}+\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}-2\hat{j}+3\hat{k}).$$

Sol. The equation of the plane is $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the equation is $(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - \hat{j}) = \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$

Thus, vectors $(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - \hat{j}), \hat{i} + \hat{j} + \hat{k}, \hat{i} - 2\hat{j} + 3\hat{k}$ are coplanar.

Therefore, the equation of the plane is $\begin{vmatrix} x-1 & y-(-1) & z-0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 0$ or $5x - 2y - 3z - 7 = 0$

EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND LINE

Let the plane pass through a given point $A(\vec{a})$ and line $\vec{r} = \vec{b} + \lambda\vec{c}$.

For any position of point $R(\vec{r})$ on the plane, vectors $\overrightarrow{AB}$, $\overrightarrow{RA}$ and $\vec{c}$ are coplanar. Then $[\vec{r} - \vec{a} \;\; \vec{b} - \vec{a} \;\; \vec{c}] = 0$, which is required equation of the plane.

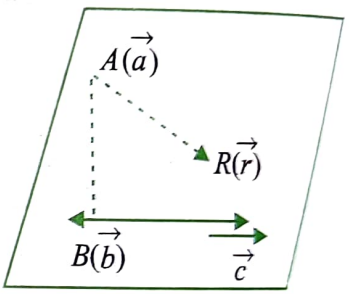


ILLUSTRATION 4.40

Prove that the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$ contains the line $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$.

Sol. To show that $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$ (i)

lies in the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$, (ii)

we must show that each point of (i) lies in (ii). In other words, we must show that $\vec{r}$ in (i) satisfies (ii) for every value of λ. We have

$$[\hat{i} + \hat{j} + \lambda(2\hat{i} + \hat{j} + 4\hat{k})] \cdot (\hat{i} + 2\hat{j} - \hat{k})$$
$$= (\hat{i} + \hat{j}) \cdot (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + 4\hat{k}) \cdot (\hat{i} + 2\hat{j} - \hat{k})$$
$$= (1)(1) + (1)(2) + \lambda[(2)(1) + (1)(2) + 4(-1)] = 3 + \lambda(0) = 3$$

Hence, line (i) lies in plane (ii).

ILLUSTRATION 4.41

Find the equation of the plane which is parallel to the lines $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$ and $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and is passing through the point (0, 1, –1).

Sol. The plane is parallel to the given lines, which are directed along vectors $\vec{a} = 2\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = -3\hat{i} + 2\hat{j} + 1\hat{k}$.

Then the plane is normal to vector $\vec{a} \times \vec{b} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 4 \\ -3 & 2 & 1 \end{vmatrix} = -7\hat{i} - 14\hat{j} + 7\hat{k}$

Also, the plane passes through the point $(0, 1, -1)$.
Therefore, the equation of the plane is $-7(x-0) - 14(y-1) + 7(z+1) = 0$ or $7x + 14y - 7z = 21$

INTERCEPT FORM OF A PLANE

Let O be the origin and let OX, OY and OZ be the coordinate axes.
Let the plane meet the coordinate axes at the points A, B and C, respectively, such that $OA = a$, $OB = b$ and $OC = c$. Then, the coordinates of points are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.
Let the equation of the plane be $Ax + By + Cz + D = 0$ (i)
Since (i) passes through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, we have

$$Aa + D = 0 \text{ or } A = \frac{-D}{a};\ Bb + D = 0 \text{ or } B = \frac{-D}{b};\ Cc + D = 0 \text{ or } C = \frac{-D}{c}$$

Putting these values in (i), we get the required equation of the plane as

$$\frac{-D}{a}x - \frac{D}{b}y - \frac{D}{c}z = -D \text{ or } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

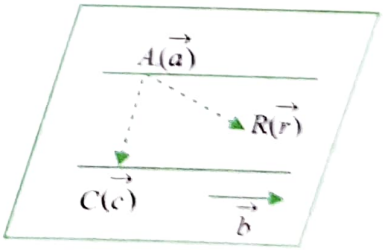
ILLUSTRATION 4.42

If a plane meets the coordinate axes at A, B and C such that the centroid of the triangle is $(1, 2, 4)$, then find the equation of the plane.

Sol. Let the plane meet the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$, and $C(0, 0, c)$. Then, $a = 3, b = 6, c = 12$.
Hence, the equation of required plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{12} = 1$ or $4x + 2y + z = 12$

EQUATION OF A PLANE PASSING THROUGH TWO PARALLEL LINES

Let the plane pass through parallel lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{b}$.



As shown in the diagram, for any position of R in the plane, vectors $\overrightarrow{RA}$, $\overrightarrow{AC}$ and $\vec{b}$ are coplanar. Then $[\vec{r} - \vec{a}\ \vec{c} - \vec{a}\ \vec{b}] = 0$, which is the required equation of the plane.

EQUATION OF A PLANE PARALLEL TO A GIVEN PLANE

The general equation of the plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is any scalar, as normal to both the planes is $a\hat{i} + b\hat{j} + c\hat{k}$.

ILLUSTRATION 4.43

Find the equation of the plane passing through (3, 4, −1), which is parallel to the plane $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+7=0$.

Sol. The equation of any plane which is parallel to $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+7=0$ is

$$\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+\lambda=0$$

or
$$2x-3y+5z+\lambda=0 \quad \text{(i)}$$

Further (i) will pass through (3, 4, −1) if (2) (3) + (−3) (4) + 5 (−1) + λ = 0 or −11 + λ = 0 or λ = 11

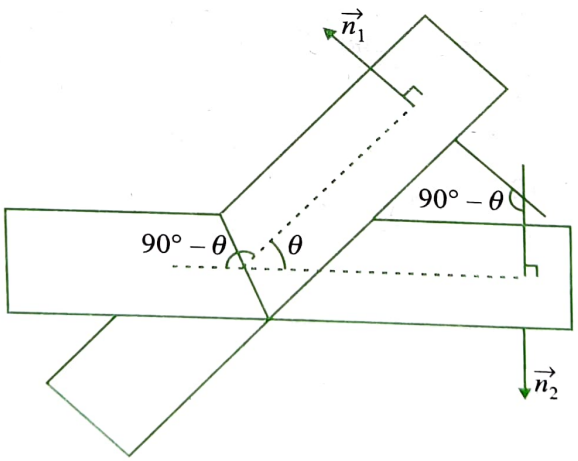
Thus, equation of the required plane is $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$.

ANGLE BETWEEN TWO PLANES

The angle between two planes is defined as the angle between their normals.

Let θ be the angle between planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$. Then

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$



Condition for Perpendicularity

If the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ are perpendicular, then $\vec{n}_1$ and $\vec{n}_2$ are perpendicular. Therefore,

$$\vec{n}_1\cdot\vec{n}_2=0$$

Condition for Parallelism

If the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ are parallel, there exists the scalar λ such that $\vec{n}_1=\lambda\vec{n}_2$.

Cartesian Form

If the planes are $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition for parallelism: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$

Condition for perpendicularity: $a_1a_2 + b_1b_2 + c_1c_2 = 0$

ILLUSTRATION 4.44

Find the angle between the planes $2x + y - 2x + 3 = 0$ and $\vec{r} \cdot (6\hat{i} + 3\hat{j} + 2\hat{k}) = 5$.

Sol. Normals along the given planes are $2\hat{i} + \hat{j} - 2\hat{k}$ and $6\hat{i} + 3\hat{j} + 2\hat{k}$.

Then angle between planes, $\theta = \cos^{-1}\dfrac{(2\hat{i} + \hat{j} - 2\hat{k}) \cdot (6\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{(2)^2 + (1)^2 + (-2)^2}\sqrt{(6)^2 + (3)^2 + (2)^2}} = \cos^{-1}\dfrac{11}{21}$

ILLUSTRATION 4.45

Show that $ax + by + r = 0$, $by + cz + p = 0$ and $cz + ax + q = 0$ are perpendicular to x–y, y–z and z–x planes, respectively.

Sol. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular to each other if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

The equation of x–y, y–z and z–x planes are $z = 0$, $x = 0$ and $y = 0$, respectively.

Now we have to show that $z = 0$ is perpendicular to $ax + by + r = 0$.

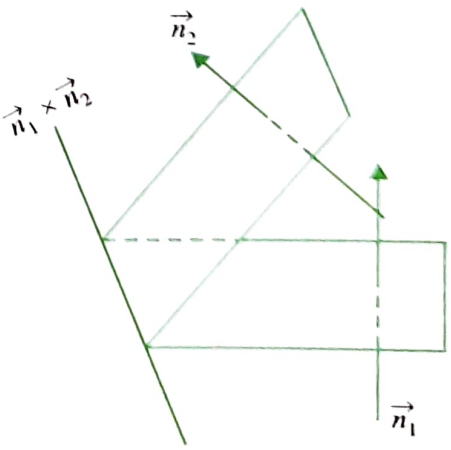
It follows immediately, since $a\,(0) + b\,(0) + (0)\,(1) = 0$, other parts can be done similarly.

LINE OF INTERSECTION OF TWO PLANES

Let two non-parallel planes be $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$

Now line of intersection of planes is perpendicular to vectors $\vec{n}_1$ and $\vec{n}_2$.

Therefore, line of intersection is parallel to vector $\vec{n}_1 \times \vec{n}_2$.



If we wish to find the equation of line of intersection of planes $a_1x + b_1y + c_1z - d_1 = 0$ and $a_2x + b_2y + c_2z - d_2 = 0$, then we find any point on the line by putting $z = 0$ (say), then we can find corresponding values of x and y by solving equations $a_1x + b_1y - d_1 = 0$ and $a_2x + b_2y - d_2 = 0$. Thus, by fixing the value of $z = \lambda$, we can find the corresponding value of x and y in terms of λ. After getting x, y and z in terms of λ, we can find the equation of line in symmetric form.

ILLUSTRATION 4.46

Reduce the equation of line $x - y + 2z = 5$ and $3x + y + z = 6$ in symmetrical form.

or

Find the line of intersection of planes $x - y + 2z = 5$ and $3x + y + z = 6$.

Sol. Given $x - y + 2z = 5$, $3x + y + z = 6$.

Let $z = \lambda$

Then $x - y = 5 - 2\lambda$ and $3x + y = 6 - \lambda$.

Solving these two equations, $4x = 11 - 3\lambda$ and $4y = 4x - 20 + 8\lambda = -9 + 5\lambda$.

The equation of the line is $\frac{4x-11}{-3} = \frac{4y+9}{5} = \frac{z-0}{1}$.

ILLUSTRATION 4.47

Find the angle between the lines $x - 3y - 4 = 0$, $4y - z + 5 = 0$ and $x + 3y - 11 = 0$, $2y - z + 6 = 0$.

Sol. Line $x - 3y - 4 = 0 = 4y - z + 5$ is line of intersection of planes $x - 3y - 4 = 0$ and $4y - z + 5 = 0$.

Vector along the line of intersection is $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 0 \\ 0 & 4 & -1 \end{vmatrix} = 3\hat{i} + \hat{j} + 4\hat{k} = \vec{a}$

Line $x + 3y - 11 = 0 = 2y - z + 6$ is line of intersection of planes $x + 3y - 11 = 0$ and $2y - z + 6 = 0$.

Vector along line of intersection is $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 0 \\ 0 & 2 & -1 \end{vmatrix} = -3\hat{i} + \hat{j} + 2\hat{k} = \vec{b}$

$$\vec{a} \cdot \vec{b} = (3)(-3) + (1)(1) + (4)(2) = 0$$

Therefore, lines are perpendicular.

ILLUSTRATION 4.48

If the line $x = y = z$ intersect the line $\sin A \cdot x + \sin B \cdot y + \sin C \cdot z = 2d^2$, $\sin 2A \cdot x + \sin 2B \cdot y + \sin 2C \cdot z = d^2$, then find the value of $\sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}$, where A, B, C are the angles of a triangle.

Sol. Any point on the line $x = y = z$ is $(\lambda, \lambda, \lambda)$

This point lies on the planes

$$\sin A \cdot x + \sin B \cdot y + \sin C \cdot z = 2d^2$$

and $$\sin 2A \cdot x + \sin 2B \cdot y + \sin 2C \cdot z = d^2$$

$\therefore$ $$\sin A\,\lambda + \sin B\,\lambda + \sin C\,\lambda = 2d^2$$

and $$\sin 2A\,\lambda + \sin 2B\,\lambda + \sin 2C\,\lambda = d^2$$

Eliminating d^2, we get

$$\sin A + \sin B + \sin C = 2(\sin 2A + \sin 2B \cdot + \sin 2C)$$

$$4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=8\sin A\sin B\sin C$$

or

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{16}$$

or

ILLUSTRATION 4.49

Find the point of intersection of line passing through (0, 0, 1) and intersecting lines $x+2y+z=1$, $-x+y-2z=2$ and $x+y=2$, $x+z=2$ with the xy plane.

Sol. Equation of line is

$$x+2y+z-1+\lambda(-x+y-2z-2)=0$$

and

$$x+y-2+\mu(x+z-2)=0$$

This line passes through the point (0, 0, 1). Thus,

$$\lambda=0,\ \mu=-2$$

Hence, equation of line is $x+2y+z-1=0=-x+y-2z+2$

For its point of intersection with xy plane, put $z=0$.

$\therefore$ $x+2y=1,\ -x+y+2=0$

Solving we get point of intersection as $\left(\frac{5}{3},\frac{-1}{3},0\right)$

ILLUSTRATION 4.50

A horizontal plane $4x-3y+7z=0$ is given. Find a line of greatest slope passes through the point (2, 1, 1) in the plane $2x+y-5z=0$.

Sol. The required line through the point $P(2, 1, 1)$ in the plane $2x+y-5z=0$ and of greatest slope is perpendicular to line of intersection of the planes

$$2x+y-5z=0 \text{ and } 4x-3y+7z=0$$

Vector along line of intersection of planes is

$$\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&1&-5\\4&-3&7\end{vmatrix}=-8\hat{i}-34\hat{j}-10\hat{k}$$

Then equation of line is $\frac{x-2}{-8}=\frac{y-1}{-34}=\frac{z-1}{-10}$ or $\frac{x-2}{4}=\frac{y-1}{17}=\frac{z-1}{5}$

ILLUSTRATION 4.51

Find the equation of the plane passing through the points (–1, 1, 1) and (1, –1, 1) and perpendicular to the plane $x+2y+2z=5$.

Sol. The equation of any plane which passes through (–1, 1, 1) is

$$a(x+1)+b(y-1)+c(z-1)=0 \qquad \text{(i)}$$

This plane will pass through $(1, -1, 1)$ if

$$2a - 2b = 0 \text{ or } a = b \quad \text{(ii)}$$

Next, (i) will be perpendicular to $x + 2y + 2z = 5$ if

$$a + 2b + 2c = 0 \quad \text{(iii)}$$

Using (ii), we can write (iii) as $a + 2a + 2c = 0$ or $c = -3a/2$. Thus,

$$a : b : c = a : a : \left(\frac{-3}{2}\right)a = 2 : 2 : -3$$

Putting these values in (i), we get $2(x + 1) + 2(y - 1) - 3(z - 1) = 0$ or $2x + 2y - 3z + 3 = 0$, which is the equation of the required plane.

Alternate method:

The plane is passing through the points $A(-1, 1, 1)$ and $B(1, -1, 1)$.

Let any point on the plane be $P(x, y, z)$.

Then vector $\overrightarrow{AP} \times \overrightarrow{AB}$ is perpendicular to vector $\hat{i} + 2\hat{j} + 2\hat{k}$, which is normal to the plane $x + 2y + 2z = 5$.

Hence, the equation of the plane is $\begin{vmatrix} x-(-1) & y-1 & z-1 \\ 1-(-1) & -1-1 & 1-1 \\ 1 & 2 & 2 \end{vmatrix} = 0$ or $2x + 2y - 3z + 3 = 0$

ILLUSTRATION 4.52

Find the equation of the plane containing line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and point $(0, 7, -7)$.

Sol. The equation of the plane containing line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ is

$$a(x + 1) + b(y - 3) + c(z + 2) = 0, \quad \text{(i)}$$

where $$-3a + 2b + c = 0 \quad \text{(ii)}$$

This passes through $(0, 7, -7)$. Thus,

$$a + 4b - 5c = 0 \quad \text{(iii)}$$

From (ii) and (iii), $\frac{a}{-14} = \frac{b}{-14} = \frac{c}{-14}$ or $\frac{a}{1} = \frac{b}{1} = \frac{c}{1}$

So, the required plane is $x + y + z = 0$.

ILLUSTRATION 4.53

Find the distance of the point $P(3, 8, 2)$ from the line $\frac{1}{2}(x - 1) = \frac{1}{4}(y - 3) = \frac{1}{3}(z - 2)$ measured parallel to the plane $3x + 2y - 2z + 15 = 0$.

Sol. Let the general point of the line be $A(2\lambda + 1, 4\lambda + 3, 3\lambda + 2)$.

Let this point lie on the line such that AP is parallel to the plane. Then

$$\overrightarrow{AP} \perp (3\hat{i} + 2\hat{j} - 2\hat{k})$$

$$3\cdot(2\lambda-2)+2(4\lambda-5)-2(3\lambda)=0$$

or $\lambda=2$

or Therefore, A is $(5, 11, 8)$.

$$PA=\sqrt{(5-3)^2+(11-8)^2+(8-2)^2}$$

$$=\sqrt{4+9+36}=7$$

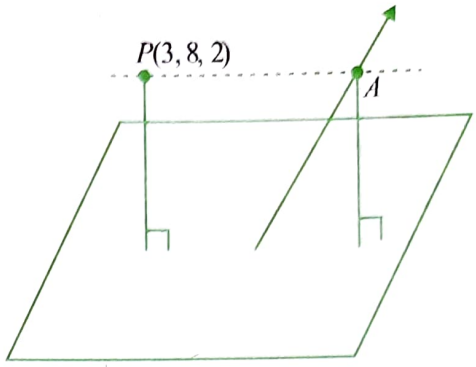


ILLUSTRATION 4.54

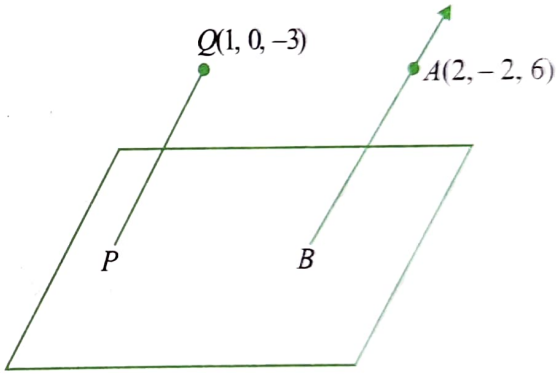
Find the distance of the point $(1, 0, -3)$ from the plane $x-y-z=9$ measured parallel to the line $\frac{x-2}{2}=\frac{y+2}{3}=\frac{z-6}{-6}$.

Sol. The given plane is $x-y-z=9$ (i)

The given line AB is $\frac{x-2}{2}=\frac{y+2}{3}=\frac{z-6}{-6}$ (ii)

The equation of the line passing through $(1, 0, -3)$ and parallel to $\frac{x-2}{2}=\frac{y+2}{3}=\frac{z-6}{-6}$ is

$$\frac{x-1}{2}=\frac{y-0}{3}=\frac{z+3}{-6}=r \quad \text{(iii)}$$



Coordinate of any point on (iii) may be given as $P(2r+1, 3r, -6r-3)$.

If P is the point of the intersection of (i) and (iii), then it must lie on (i). Therefore,

$$(2r+1)-(3r)-(-6r-3)=9$$

$$2r+1-3r+6r+3=9 \text{ or } r=1$$

Therefore, the coordinates of P are 3, 3, –9.

Distance between $Q(1, 0, -3)$ and $P(3, 3, -9)$ $=\sqrt{(3-1)^2+(3-0)^2+(-9+3)^2}$

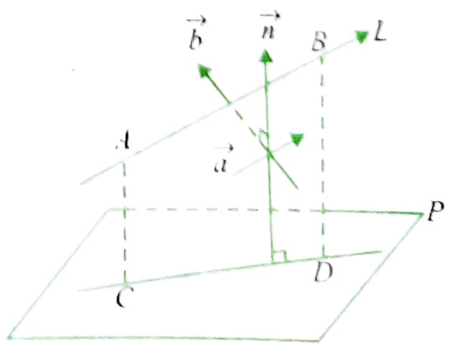
$$=\sqrt{4+9+36}=7$$

ILLUSTRATION 4.55

Find the equation of the projection of the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ on the plane $x+2y+z=9$.

Sol. Projection of given line $L\equiv\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ on given plane $P\equiv x+2y+z=9$ is the line of intersection of plane P and plane $(ABCD)$ through the line L and perpendicular to the plane P.

Now line L is parallel to vector $\vec{a}=2\hat{i}-\hat{j}+4\hat{k}$ (i)



Vector normal to plane P is $\vec{n} = \hat{i} + 2\hat{j} + \hat{k}$ (ii)

$\therefore$ Vector normal plane $ABCD$ is $\vec{b} = \vec{a} \times \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = -9\hat{i} + 2\hat{j} + 5\hat{k}$

Plane $ABCD$ passes through the line L, i.e., passes through the point $(1, -1, 3)$. Thus,

Equation of plane $ABCD = (-9)(x - 1) + 2(y + 1) + 5(z - 3) = 0$ or $9x - 2y - 5z + 4 = 0$ (iii)

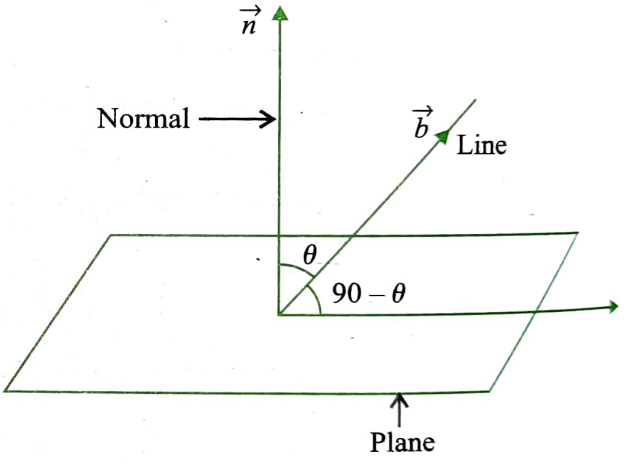
Now required projection is line of intersection of given plane P and plane $ABCD$

or $x + 2y + z - 9 = 9x - 2y - 5z + 4$

ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

If the equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$ and that of the plane is $\vec{r} \cdot \vec{n} = d$, then angle θ between the line and the normal to the plane is $\cos\theta = \left|\dfrac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}\right|$.



So the angle ϕ between the line and the plane is given by $90° - \theta$. Thus,

$$\sin\phi = \left|\frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}\right| \text{ or } \phi = \sin^{-1}\left|\frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}\right|$$

Line $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r} \cdot \vec{n} = d$ are perpendicular if $\vec{b} = \lambda\vec{n}$ or $\vec{b} \times \vec{n} = \vec{0}$ and parallel if $\vec{b} \perp \vec{n}$ or $\vec{b} \cdot \vec{n} = 0$.

ILLUSTRATION 4.56

Find the angle between the line $\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 4$.

Sol. We know that if θ is the angle between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} \cdot \vec{n} = p$, then $\sin\theta = \left|\dfrac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}\right|$

Therefore, if θ is the angle between $\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 4$, then

$$\sin\theta = \left|\frac{(\hat{i} - \hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})}{|\hat{i} - \hat{j} + \hat{k}||2\hat{i} - \hat{j} + \hat{k}|}\right|$$

$$= \frac{2 + 1 + 1}{\sqrt{1 + 1 + 1}\sqrt{4 + 1 + 1}}$$

$$= \frac{4}{\sqrt{3}\sqrt{6}} = \frac{4}{3\sqrt{2}}$$

or $$\theta = \sin^{-1}\left(\frac{4}{3\sqrt{2}}\right)$$

ILLUSTRATION 4.57

Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}+\hat{j}+\hat{k})=6$.

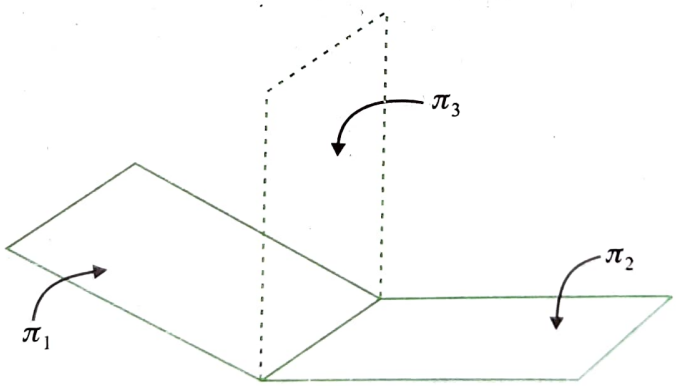
Sol. Line is parallel to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}+\hat{j}+\hat{k})=6$, i.e., line is perpendicular to the vectors $\hat{i}-\hat{j}+2\hat{k}$ and $3\hat{i}+\hat{j}+\hat{k}$

$\therefore$ Line is parallel to the vector $(\hat{i}-\hat{j}+2\hat{k})\times(3\hat{i}+\hat{j}+\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{vmatrix} = -3\hat{i}+5\hat{j}+4\hat{k}$

Line passes through (1, 2, 3).

$\therefore$ Equation of line is $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k}) + \lambda(-3\hat{i}+5\hat{j}+4\hat{k})$

EQUATION OF A PLANE PASSING THROUGH THE LINE OF INTERSECTION OF TWO PLANES



Let π_1 and π_2 be two planes with equations $\vec{r}\cdot\hat{n_1} = d_1$ and $\vec{r}\cdot\hat{n_2} = d_2$, respectively. The position vector of any point on the line of intersection must satisfy both the equations.

If $\vec{t}$ is the position vector of a point on the line, then

$$\vec{t}\cdot\hat{n_1} = d_1 \text{ and } \vec{t}\cdot\hat{n_2} = d_2$$

Therefore, for all real values of λ, we have

$$\vec{t}\cdot(\hat{n_1}+\lambda\hat{n_2}) = d_1 + \lambda d_2 \qquad \text{(i)}$$

Since $\vec{t}$ is arbitrary, it satisfies for any point on the line.

Hence, the equation $\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2}) = d_1+\lambda d_2$ represents a plane π_3 which is such that if any vector $\vec{r}$ satisfies both the equations π_1 and π_2, it also satisfies the equation π_3.

Cartesian Form

In Cartesian system, let $\vec{n_1} = A_1\hat{i} + B_1\hat{j} + C_1\hat{k}$, $\vec{n_2} = A_2\hat{i} + B_2\hat{j} + C_2\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
Then (i) becomes

$$x(A_1 + \lambda A_2) + y(B_1 + \lambda B_2) + z(C_1 + \lambda C_2) = d_1 + \lambda d_2$$

or
$$(A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0 \quad \text{(ii)}$$

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

ILLUSTRATION 4.58

Find the equation of the plane which contains the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$ and $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$.

Sol. The equation of the given planes are

$$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0 \quad \text{(i)}$$

$$\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0 \quad \text{(ii)}$$

The equation of the plane passing through the line of intersection of the plane given in (i) and (ii) is

$$[\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4] + \lambda[\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5] = 0$$

$$\vec{r} \cdot [(2\lambda + 1)\hat{i} + (\lambda + 2)\hat{j} + (3 - \lambda)\hat{k}] + (5\lambda - 4) = 0 \quad \text{(iii)}$$

The plane in (iii) is perpendicular to the plane, $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$. Thus,

$$5(2\lambda + 1) + 3(\lambda + 2) - 6(3 - \lambda) = 0$$

or
$$19\lambda - 7 = 0$$

or
$$\lambda = \frac{7}{19}$$

Substituting $\lambda = \frac{7}{19}$ in (iii), we get

$$\vec{r} \cdot \left[\frac{33}{19}\hat{i} + \frac{45}{19}\hat{j} + \frac{50}{19}\hat{k}\right] - \frac{41}{19} = 0$$

or
$$\vec{r} \cdot (33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 = 0$$

or
$$33x + 45y + 50z - 41 = 0$$

ILLUSTRATION 4.59

Find the equation of a plane containing the line of intersection of the planes $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ and passing through (1, 1, 1).

Sol. The equation of a plane passing through the line of intersection of the given planes is

$$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0 \quad \text{(i)}$$

If it is passes through (1, 1, 1), $(1 + 1 + 1 - 6) + \lambda(2 + 3 + 4 + 5) = 0$, then

$$\lambda = \frac{3}{14}$$

Putting $\lambda = 3/14$ in (i), we get

$$(x + y + z - 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0$$

$$20x + 23y + 26z - 69 = 0$$

ILLUSTRATION 4.60

The plane $ax + by = 0$ is rotated through an angle α about its line of intersection with the plane $z = 0$. Show that the equation to the plane in the new position is $ax + by \pm z\sqrt{a^2 + b^2}\tan\alpha = 0$.

Sol. Given planes are $ax + by = 0$ (i)

and $z = 0$ (ii)

Therefore, the equation of any plane passing through the line of intersection of planes (i) and (ii) may be taken as

$$ax + by + kz = 0 \quad \text{(iii)}$$

The direction cosines of a normal to the plane (iii) are

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}, \frac{b}{\sqrt{a^2 + b^2 + k^2}} \text{ and } \frac{k}{\sqrt{a^2 + b^2 + k^2}}$$

The direction cosines of a normal to the plane (i) are

$$\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \text{ and } 0$$

Since the angle between the planes (i) and (iii) is α, we have

$$\cos\alpha = \frac{a \cdot a + b \cdot b + k \cdot 0}{\sqrt{a^2 + b^2 + k^2}\sqrt{a^2 + b^2}} = \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + k^2}}$$

or $$k^2\cos^2\alpha = a^2(1 - \cos^2\alpha) + b^2(1 - \cos^2\alpha)$$

or $$k^2 = \frac{(a^2 + b^2)\sin^2\alpha}{\cos^2\alpha} \text{ or } k = \pm\sqrt{a^2 + b^2}\tan\alpha$$

Putting this in (iii), we get the equation of the plane as $ax + by \pm z\sqrt{a^2 + b^2}\tan\alpha = 0$

DISTANCE OF A POINT FROM A PLANE

Vector Form

Let $\pi\ (\vec{r} \cdot \vec{n} = d)$ be the given plane and $P\ (\vec{a})$ be the given point.

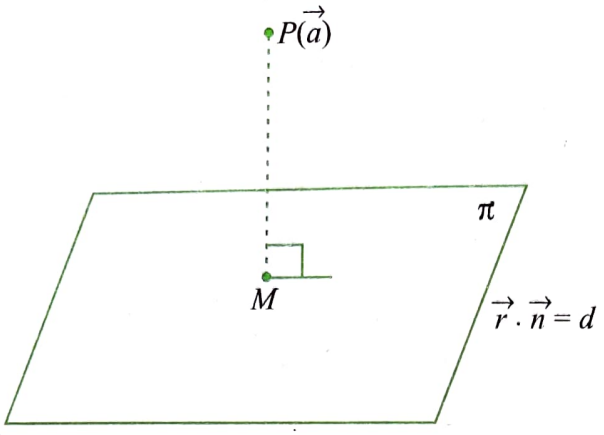
Let PM be the length of the perpendicular from P to the plane π.

Since line PM passes through $P(\vec{a})$ and is parallel to vector $\vec{n}$, which is normal to the plane π,

the vector equation of line PM is $\vec{r} = \vec{a} + \lambda\vec{n}$ (i)

Point M is the intersection of (i) and the given plane π. Therefore,

$$(\vec{a} + \lambda\vec{n}) \cdot \vec{n} = d$$



or $\vec{a}\cdot\vec{n}+\lambda\vec{n}\cdot\vec{n}=d$

or $\lambda=\dfrac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}$

Putting the value of λ in (i), we obtain the position vector of M given by $\vec{r}=\vec{a}+\left(\dfrac{d-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\right)\vec{n}$

$$\overrightarrow{PM}=\text{P.V. of } M-\text{P.V. of } P$$

$$=\vec{a}+\left(\frac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}-\vec{a}$$

$$=\left(\frac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}$$

$\Rightarrow$ $PM=|\overrightarrow{PM}|=\left|\dfrac{(d-\vec{a}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}\right|=\dfrac{|d-(\vec{a}\cdot\vec{n})||\vec{n}|}{|\vec{n}|^2}=\dfrac{|d-(\vec{a}\cdot\vec{n})|}{|\vec{n}|}$, which is the required length.

Cartesian Form

Let PM be the length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Then the equation of PM is $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}=r$ (let) (i)

The coordinates of any point on this line are $(x_1 + ar, y_1 + br, z_1 + cr)$.

Thus, the point coincides with M if and only if it lies on the plane, i.e.,

$$a(x_1+ar)+b(y_1+br)+c(z_1+cr)+d=0$$

$$r=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2} \quad \text{(ii)}$$

Now, $PM=\sqrt{(x_1+ar-x_1)^2+(y_1+br-y_1)^2+(z_1+cr-z_1)^2}$

$$=\sqrt{(a^2+b^2+c^2)r^2}$$

$$= \sqrt{a^2+b^2+c^2}\ |r|$$

$$= \sqrt{a^2+b^2+c^2}\left|\frac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}\right| \qquad \text{[from (ii)]}$$

$$= \frac{|(ax_1+by_1+cz_1+d)|}{\sqrt{a^2+b^2+c^2}}$$

Also, if coordinates of M are (x_2, y_2, z_2), then

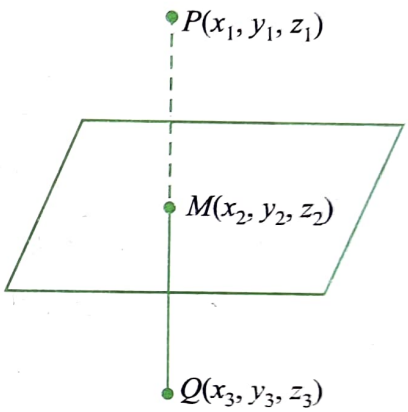
$$\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = \frac{z_2-z_1}{c} = -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2} \qquad \text{(iii)}$$

Image of a Point in a Plane

Here Q is the image of P in the plane.

Therefore, M is the midpoint of PQ.

Therefore, from (iii), we have



$$\frac{\frac{x_3+x_1}{2}-x_1}{a} = \frac{\frac{y_3-y_1}{2}-y_1}{b} = \frac{\frac{z_3-z_1}{2}-z_1}{c}$$

$$= -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

or

$$\frac{x_3-x_1}{a} = \frac{y_3-y_1}{b} = \frac{z_3-z_1}{c} = \frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

Distance Between Parallel Planes

The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$d = \left|\frac{(d_2-d_1)}{\sqrt{a^2+b^2+c^2}}\right|$$

Proof:

Let $P(x_1, y_1, z_1)$ be point on plane $ax + by + cz + d_1 = 0$.

Then distance of this point from plane $ax + by + cz + d_2 = 0$ is

$$d = \frac{|ax_1+by_1+cz_1+d_2|}{\sqrt{a^2+b^2+c^2}}$$

Also

$$ax_1 + by_1 + cz_1 + d_1 = 0$$

$\Rightarrow$

$$d = \frac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$$

Illustration 4.61

Find the length and the foot of the perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$.

Sol. The required length $= \frac{2(7)+4(14)-(5)-2}{\sqrt{2^2+4^2+1^2}} = \frac{14+56-5-2}{\sqrt{4+16+1}} = \frac{63}{\sqrt{21}}$

Let the coordinates of the foot of the perpendicular from the point $P(7, 14, 5)$ be $M(\alpha, \beta, \gamma)$.
Then the direction ratios of PM are $\alpha - 7, \beta - 14$ and $\gamma - 5$.
Therefore, the direction ratios of the normal to the plane are $\alpha - 7, \beta - 14$ and $\gamma - 5$.
But the direction ratios of normal to the given plane $2x + 4y - z = 2$ are 2, 4 and -1. Hence,

$$\frac{\alpha - 7}{2} = \frac{\beta - 14}{4} = \frac{\gamma - 5}{-1} = k$$

$\therefore$ $\alpha = 2k + 7, \beta = 4k + 14$ and $\gamma = -k + 5$ (i)

Since α, β and γ lie on the plane $2x + 4y - z = 2$, $2\alpha + 4\beta - \gamma = 2$

or $2(7 + 2k) + 4(14 + 4k) - (5 - k) = 2$

or $14 + 4k + 56 + 16k - 5 + k = 2$

or $21k = -63$

or $k = -3$

Now, putting $k = -3$ in (i), we get

$$\alpha = 1, \beta = 2, \gamma = 8$$

Hence, the foot of the perpendicular is $(1, 2, 8)$.

ILLUSTRATION 4.62

Find the locus of a point, the sum of squares of whose distances from the planes $x - z = 0, x - 2y + z = 0$ and $x + y + z = 0$ is 36.

Sol. Given planes are

$$x - z = 0, x - 2y + z = 0, x + y + z = 0$$

Let the point whose locus is required be $P(\alpha, \beta, \gamma)$.
According to question,

$$\frac{|\alpha - \gamma|^2}{2} + \frac{|\alpha - 2\beta + \gamma|^2}{6} + \frac{|\alpha + \beta + \gamma|^2}{3} = 36$$

or $3(\alpha^2 + \gamma^2 - 2\alpha\gamma) + \alpha^2 + 4\beta^2 + \gamma^2 - 4\alpha\beta - 4\beta\gamma + 2\alpha\gamma$
$+ 2(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma) = 36 \times 6$

or $6\alpha^2 + 6\beta^2 + 6\gamma^2 = 36 \times 6$

or $\alpha^2 + \beta^2 + \gamma^2 = 36$

Hence, the required equation of locus is $x^2 + y^2 + z^2 = 36$.

ILLUSTRATION 4.63

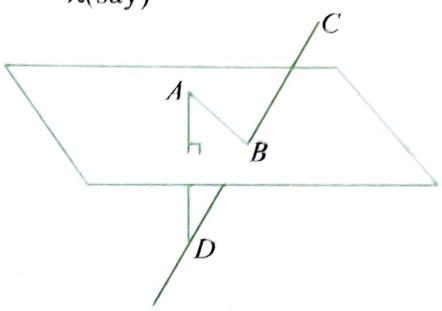
A ray of light passing through the point $A(1,2,3)$, strikes the plane $x + y + z = 12$ at B and on reflection passes through point $C(3, 5, 9)$. Find the coordinates of point B.

Sol. Let the image of point $A(1, 2, 3)$ about $x + y + z = 12$ be $D(p, q, r)$. Then

$$\frac{p - 1}{1} = \frac{q - 2}{1} = \frac{z - 3}{1} = \frac{-2(1 + 2 + 3 - 12)}{1^2 + 1^2 + 1^2}$$

$\therefore$ $D(p, q, r) \equiv D(5, 6, 7)$

Now equation of line CD is $\frac{x-5}{3-5}=\frac{y-6}{5-6}=\frac{z-7}{9-7}$ or $\frac{x-5}{-2}=\frac{y-6}{-1}=\frac{z-7}{2}=\lambda$(say)



Point B is point of intersection of AC with the plane.

Let $B \equiv (-2\lambda+5, -\lambda+6, 2\lambda+7)$

This point lies on the plane. Therefore,

$-2\lambda+5-\lambda+6+2\lambda+7=12$

or $\lambda=6$

$\therefore$ $B \equiv (-7, 0, 19)$

ILLUSTRATION 4.64

Find the distance between the parallel planes $x+2y-2z+1=0$ and $2x+4y-4z+5=0$.

Sol. We know that the distance between parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is

$$d=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

Therefore, the distance between $x+2y-2z+1=0$ and $x+2y-2z+\frac{5}{2}=0$ is

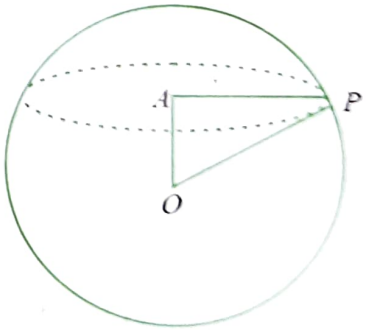
$$\frac{|(5/2)-1|}{\sqrt{1+4+4}}=\frac{1}{2}$$

ILLUSTRATION 4.65

Find the radius of the circular section in which the sphere $|\vec{r}|=5$ is cut by the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=3\sqrt{3}$.

Sol. Let A be the foot of the perpendicular from the centre O to the plane

$$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-3\sqrt{3}=0$$



Then $$|OA|=\left|\frac{0\cdot(\hat{i}+\hat{j}+\hat{k})-3\sqrt{3}}{|\hat{i}+\hat{j}+\hat{k}|}\right|=\frac{3\sqrt{3}}{\sqrt{3}}=3$$

(Perpendicular distance of a point from the plane)

If P is any point on the circle, then P lies on the plane as well as on the sphere. Therefore,

OP = Radius of the sphere = 5

Now $AP^2=OP^2-OA^2=5^2-3^2=16$ or $AP=4$

ILLUSTRATION 4.66

Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$.

Sol. The given plane will touch the given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere. The centre of the given sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ is $(1, 2, -1)$ and its radius is $\sqrt{1^2+2^2+(-1)^2-(-3)}=3$.

Length of the perpendicular from $(1, 2, -1)$ to the plane $2x - 2y + z + 12 = 0$ is

$$\left|\frac{2(1)-2(2)+(-1)+12}{\sqrt{2^2+(-2)^2+1^2}}\right| = \frac{9}{3} = 3$$

Thus, the given plane touches the given sphere.

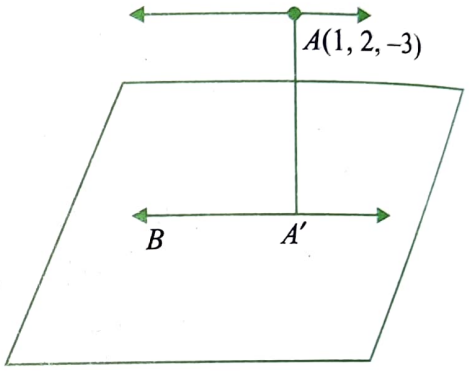
ILLUSTRATION 4.67

Find the image of the line $\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3}$ in the plane $3x - 3y + 10z - 26 = 0$.

Sol.

$$\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3} \quad \text{(i)}$$

$$3x - 3y + 10z - 26 = 0 \quad \text{(ii)}$$



The direction ratios of the line are 9, –1 and –3 and direction ratios of the normal to the given plane are 3, –3 and 10.

Since $9 \cdot 3 + (-1)(-3) + (-3)\,10 = 0$ and the point $(1, 2, -3)$ of line (i) does not lie in plane (ii) for $3 \cdot 1 - 3 \cdot 2 + 10 \cdot (-3) - 26 \neq 0$, line (i) is parallel to plane (ii). Let A' be the image of point $A\ (1, 2, -3)$ in plane (ii). Then the image of the line (i) in the plane (ii) is the line through A' and parallel to the line (i).

Let point A' be (p, q, r). Then

$$\frac{p-1}{3}=\frac{q-2}{-3}=\frac{r+3}{10}=-\frac{(3(1)-3(2)+10(-3)-26)}{9+9+100}=\frac{1}{2}$$

The point is $A'\ (5/2, 1/2, 2)$.

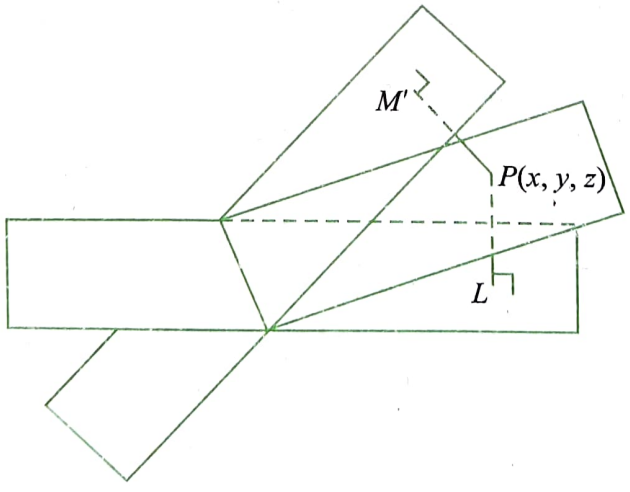
The equation of line BA' is $\frac{x-(5/2)}{9}=\frac{y-(1/2)}{-1}=\frac{z-2}{-3}$

EQUATION OF A PLANE BISECTING THE ANGLE BETWEEN TWO PLANES

Given planes are

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{(i)}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{(ii)}$$



Let $P(x, y, z)$ be a point on the plane bisecting the angle between (i) and (ii).

Let PL and PM be the length of the perpendiculars from P to planes (i) and (ii). Therefore,

$$PL = PM$$

$$\Rightarrow \quad \left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$$\text{or} \quad \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

This is the equation of the plane bisecting the angles between planes (i) and (ii).

Vector Form

The equation of the plane bisecting the angle between planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is

$$\left|\frac{\vec{r} \cdot \vec{n}_1 - d_1}{\vec{n}_1}\right| = \left|\frac{\vec{r} \cdot \vec{n}_2 - d_2}{\vec{n}_2}\right|$$

Bisector of the Angle between the Two Planes Containing the Origin

Let the equation of the two planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{(i)}$$

and

$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{(ii)}$$

where d_1 and d_2 are positive.

The equation of the bisector of the angle between the planes (i) and (ii) containing the origin is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Bisector of the Acute and Obtuse Angles between Two Planes

Let the two planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{(i)}$$

where $d_1, d_2 > 0$

$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{(ii)}$$

i. If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, the origin lies in the obtuse angle between the two planes and the equation of the bisector of the obtuse angle is $\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

ii. If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, the origin lies in the acute angle between the two planes and the equation of the bisector of the acute angle between the two planes is $\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

ILLUSTRATION 4.68

Find the equations of the bisectors of the angles between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ and specify the plane which bisects the obtuse angle.

Sol. The given planes are $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$, where $d_1, d_2 > 0$ and $a_1a_2 + b_1b_2 + c_1c_2 = 6 + 2 + 12 > 0$.

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$ (obtuse angle bisector)

$$\frac{a_1x + b_1y + c_1z + d}{\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$ (acute angle bisector)

i.e., $$\frac{2x - y + 2z + 3}{\sqrt{4+1+4}} = \pm \frac{3x - 2y + 6z + 8}{\sqrt{9+4+36}}$$

or $$(14x - 7y + 14z + 21) = \pm (9x - 6y + 18z + 24)$$

Taking the positive sign on the right hand side, we get

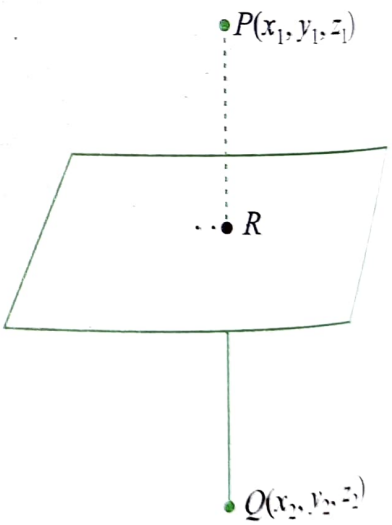
$$5x - y - 4z - 3 = 0$$ (obtuse angle bisector)

Taking the negative sign on the right hand side, we get

$$23x - 13y + 32z + 45 = 0$$ (acute angle bisector)

TWO SIDES OF A PLANE

Let $ax + by + cz + d = 0$ be the plane. Then the points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same side or the opposite sides as $\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$ > 0 or < 0, respectively.



Proof:

Here the equation of the plane is $ax + by + cz + d = 0$. (i)

Let (i) divide the line segment joining P and Q at R internally in the ratio $m : n$. Then

$$R\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

R lies on plane (i). Therefore,

$$a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c\left(\frac{mz_2 + nz_1}{m+n}\right) + d = 0$$

$$a(mx_2 + nx_1) + b(my_2 + ny_1) + c(mz_2 + nz_1) + d(m+n) = 0$$

$$m(ax_2 + by_2 + cz_2 + d) + n(ax_1 + by_1 + cz_1 + d) = 0$$

$$\frac{m}{n} = -\frac{(ax_1 + by_1 + cz_1 + d)}{(ax_2 + by_2 + cz_2 + d)}$$ (ii)

Now, if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same sign, $\frac{m}{n} < 0$ (external division)

are of opposite signs, $\frac{m}{n} > 0$ (internal division)

If
$$\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0 \quad \text{(same side)}$$

$$\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} < 0 \quad \text{(opposite side)}$$

REGULAR TETRAHEDRON

A regular tetrahedron is one in which all four faces are equilateral triangles.

ANGLE BETWEEN ANY EDGE AND FACE NOT CONTAINING THAT EDGE OF REGULAR TETRAHEDRON

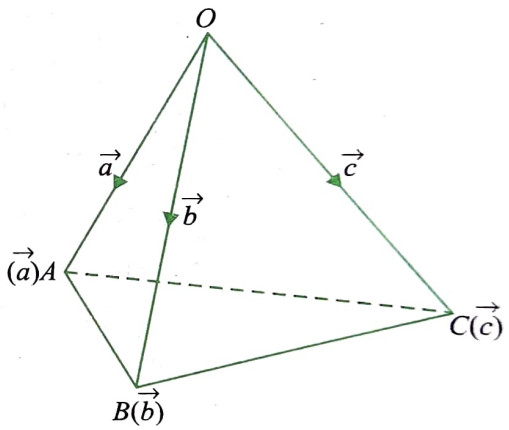
Let us find the angle between edge OA and face OBC.

Since tetrahedron is regular and its face is an equilateral triangle, the required angle is the angle between vector $\vec{a}$ and vector along angle bisector of $\angle BOC$.

Let $\vec{a}, \vec{b}, \vec{c}$ be the unit vectors.

Angle bisector of $\angle BOC$ in ΔOBC is along vector $\vec{b} + \vec{c}$.

Therefore, required angle θ is the angle between vectors $\vec{a}$ and $\vec{b} + \vec{c}$, i.e.,



$$\cos\theta = \frac{\vec{a} \cdot (\vec{b} + \vec{c})}{|\vec{a}||\vec{b} + \vec{c}|}$$

$$= \frac{\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}}{|\vec{b} + \vec{c}|}$$

Now
$$|\vec{b} + \vec{c}|^2 = b^2 + c^2 + 2\vec{b} \cdot \vec{c}$$
$$= 1 + 1 + 2(1)(1)\cos 60^\circ = 3$$

or
$$|\vec{b} + \vec{c}| = \sqrt{3}$$

$\Rightarrow$
$$\cos\theta = \frac{1 \cdot 1 \cdot \cos 60^\circ + 1 \cdot 1 \cdot \cos 60^\circ}{1 \cdot \sqrt{3}} = \frac{1}{\sqrt{3}}$$

or
$$\theta = \cos^{-1}\frac{1}{\sqrt{3}}$$

ANGLE BETWEEN THE TWO FACES OF REGULAR TETRAHEDRON

Let us find the angle between face OAB and face OAC.

Angle between faces OAB and OAC is the angle between the normal to the faces.

Normal to face OAB is $\vec{a} \times \vec{b}$; normal to face OAC is $\vec{a} \times \vec{c}$. Thus,

$$\cos\theta = \frac{(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})}{|\vec{a} \times \vec{b}| \cdot |\vec{a} \times \vec{c}|}$$

$$= \frac{\vec{a}\cdot(\vec{b}\times(\vec{a}\times\vec{c}))}{(1\cdot 1\cdot \sin 60°)^2}$$

$$= \frac{\vec{a}\,.((\vec{b}\cdot\vec{c})\,\vec{a}-(\vec{a}\cdot\vec{b})\vec{c})}{\frac{3}{4}}$$

$$= \frac{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{c})-(\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{c})}{(3/4)}$$

$$= \frac{1(1\cdot 1\cdot\cos 60°)-(1\cdot 1\cdot\cos 60°)^2}{(3/4)}$$

$$= \frac{\frac{1}{2}-\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

$\therefore$ $\theta = \cos^{-1}\frac{1}{3}$

HEIGHT OF REGULAR TETRAHEDRON

Let λ be the length of any edge of a regular tetrahedron.

Now h is the projection of OA on vector normal to the base ABC

Vector normal to the plane ABC is $(\vec{b}-\vec{a})\times(\vec{c}-\vec{a}) = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$. Therefore,

$\therefore$ $$h = \frac{\vec{a}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$$

$$= \frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{2(\text{Area of base})}$$

Now $$|\vec{a}\cdot\vec{b}| = |\vec{b}\cdot\vec{c}| = |\vec{c}\cdot\vec{a}| = \lambda^2\cos 60° = \frac{\lambda^2}{2}$$

and $$|\vec{a}\cdot\vec{a}| = |\vec{b}\cdot\vec{b}| = |\vec{c}\cdot\vec{c}| = \lambda^2$$

$$[\vec{a}\vec{b}\vec{c}\,]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = \frac{\lambda^6}{2}$$

Also $$\text{Area of } \Delta ABC = \frac{\sqrt{3}}{2}\lambda^2$$

$\therefore$ $$h = \frac{\lambda^3/\sqrt{2}}{\frac{\sqrt{3}\lambda^2}{2}} = \lambda\sqrt{\frac{2}{3}}$$

Concept Application Exercise 4.3

1. Find the angle between the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-1}{4}$ and the plane $2x + y - 3z + 4 = 0$.
2. Find the distance between the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z-2}{1}$ and the plane $x + y + z + 3 = 0$.
3. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and plane $x - y + z = 5$.
4. Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.
5. Find the equation of the plane passing through the points $(1, 0, -1)$ and $(3, 2, 2)$ and parallel to the line $x - 1 = \frac{1-y}{2} = \frac{z-2}{3}$.
6. Find the equation of a plane containing the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$.
7. Find the equation of the plane passing through the straight line $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z}{5}$ and perpendicular to the plane $x - y + z + 2 = 0$.
8. Find the equation of the plane perpendicular to the line $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{2}$ and passing through the origin.
9. Find the equation of the plane passing through the line $\frac{x-1}{5} = \frac{y+2}{6} = \frac{z-3}{4}$ and point $(4, 3, 7)$.
10. Find the angle between the line $\vec{r} = (\vec{i} + 2\vec{j} - \vec{k}) + \lambda(\vec{i} - \vec{j} + \vec{k})$ and the normal to the plane $\vec{r} \cdot (2\vec{i} - \vec{j} + \vec{k}) = 4$.
11. Find the equation of a plane which passes through the point $(1, 2, 3)$ and which is at the maximum distance from the point $(-1, 0, 2)$.
12. Find the direction ratios of orthogonal projection of line $\frac{x-1}{1} = \frac{y+1}{-2} = \frac{z-2}{3}$ in the plane $x - y + 2z - 3 = 0$. Also find the direction ratios of the image of the line in the plane.
13. Find the equation of a plane which is parallel to the plane $x - 2y + 2z = 5$ and whose distance from the point $(1, 2, 3)$ is 1.
14. Find the equation of a plane which passes through the point $(1, 2, 3)$ and which is equally inclined to the planes $x - 2y + 2z - 3 = 0$ and $8x - 4y + z - 7 = 0$.
15. Find the equation of the image of the plane $x - 2y + 2z - 3 = 0$ in the plane $x + y + z - 1 = 0$.
16. Find the equation to the plane through the points $(2, 3, 1)$ and $(4, -5, 3)$ and parallel to the x-axis.
17. Find the distance of the point $\vec{a}$ from the plane $\vec{r} \cdot \hat{n} = d$ measured parallel to the line $\vec{r} = \vec{b} + t\vec{c}$.
18. Find the value of m for which the straight line $3x - 2y + z + 3 = 0 = 4x - 3y + 4z + 1$ is parallel to the plane $2x - y + mz - 2 = 0$.

19. Prove that the lines $\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$, $\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$ are coplanar and also find the equation of the plane in which they lie. $(\alpha\gamma \neq \beta\delta)$

20. Find the plane of the intersection of $x^2+y^2+z^2+2x+2y+2z+2=0$ and $4x^2+4y^2+4z^2+4x+4y+4z-1=0$.

21. Find the radius of the circular section of the sphere $|\vec{r}|=5$ by the plane $\vec{r}\cdot(\vec{i}+2\vec{j}-\vec{k})=4\sqrt{3}$.

22. A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2+y^2+z^2-ax-by-cz=0$.

ANSWERS

1. $\sin^{-1}(-4/\sqrt{406})$ **2.** $\frac{7}{\sqrt{3}}$ **3.** 13

4. $y+z-2=0$ **5.** $4x-y-2z-6=0$ **6.** $17x-47y-24z+172=0$

7. $2x+3y+z+4=0$ **8.** $2x-y+2z=0$ **9.** $4x-8y+7z=41$

10. $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ **11.** $2x+2y+z=9$

12. D.R.s are $\frac{1}{6},\frac{1}{6},0$; D.R.s of image are $\frac{2}{3},-\frac{1}{3},1$ **13.** $x-2y+2z=6$

14. $11x-10y+7z+12=0$ **15.** $x-8y+4z-7=0$ **16.** $y+4z-7=0$

17. $\frac{|\vec{d}-\hat{a}\cdot\hat{n}|}{|\vec{c}\cdot\hat{n}|}|\vec{c}|$ **18.** $m=-2$ **19.** $x-2y+z=0$

20. $4x+4y+4z+9=0$ **21.** 3

Solved Examples

EXAMPLE 4.1

If variable lines in two adjacent positions have direction cosines l, m and n and $(l+\delta l)$, $(m+\delta m)$, $(n+\delta n)$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2=(\delta l)^2+(\delta m)^2+(\delta n)^2$.

Sol. Since l, m and n, and $(l+\delta m)$, $(m+\delta m)$, $(n+\delta n)$ are the direction cosines, we have

$$l^2+m^2+n^2=1 \quad \text{(i)}$$

$$(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1$$

or $$l^2+m^2+n^2+2l\,\delta l+2m\,\delta m+2n\,\delta n+(\delta l)^2+(\delta m)^2+(\delta n)^2=1$$

or $$2(l\,\delta l+m\,\delta m+n\,\delta n)=-(\{\delta l)^2+(\delta m)^2+(\delta n)^2\} \quad \text{(ii)}$$

Now it is given that $\delta\theta$ is the angle between two adjacent positions of the line. Therefore

$$\cos\delta\theta=l(l+\delta l)+m(m+\delta m)+n(n+\delta n) \quad \text{(iii)}$$

Now $\cos\delta\theta=1-\frac{(\delta\theta)^2}{2!}+\frac{(\delta\theta)^2}{4!}-\ldots$

If $\delta\theta$ is small, then $\cos \delta\theta = 1 - \frac{(\delta\theta)^2}{2}$

Then from (iii), we have

$$1 - \frac{(\delta\theta)^2}{2} = (l^2 + m^2 + n^2) + (l\,\delta l + m\,\delta m + n\,\delta n)$$

or
$$1 - \frac{(\delta\theta)^2}{2} = 1 - \frac{1}{2}\{(\delta l)^2 + (\delta m)^2 + (\delta n)^2\} \qquad \text{[using (i) and (ii)]}$$

or
$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

EXAMPLE 4.2

Find the equation of the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1, x = 0$, and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1, y = 0$.

Sol. The equation of the first line may be written as

$$\frac{y}{b} - \frac{1}{2} = \frac{1}{2} - \frac{z}{c}, \; x = 0$$

or
$$\frac{x}{0} = \frac{y - \frac{1}{2}b}{b} = \frac{z - \frac{1}{2}c}{-c} \qquad \text{(i)}$$

Similarly, the equation of the second line may be written as

$$\frac{x - \frac{1}{2}a}{a} = \frac{y}{0} = \frac{z + \frac{1}{2}c}{c} \qquad \text{(ii)}$$

The equation of any plane passing through line (i) is

$$A(x) + B\left(y - \frac{1}{2}b\right) + C\left(z - \frac{1}{2}c\right) = 0 \qquad \text{(iii)}$$

where
$$A \cdot 0 + B \cdot b - C \cdot c = 0 \qquad \text{(iv)}$$

Now plane (iii) will be parallel to line (ii) if

$$A \cdot a + B \cdot 0 - C \cdot c = 0 \qquad \text{(v)}$$

Solving (iv) and (v), we have $\frac{A}{bc} = \frac{B}{-ca} = \frac{C}{-ab}$

Putting these values of A, B and C in (iii), the equation of the required plane is

$$bcx - ca\left(y - \frac{1}{2}b\right) - ab\left(z - \frac{1}{2}c\right) = 0$$

or
$$\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$$

EXAMPLE 4.3

A variable plane passes through a fixed point (α, β, γ) and meets the axes at A, B and C. Show that the locus of the point of intersection of the planes through A, B and C parallel to the coordinate planes is $\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1$.

Sol. Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \tag{i}$$

where a, b and c are the parameters.

Plane (i) passes through the point (α, β, γ). Therefore,

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \tag{ii}$$

Plane (i) meets the coordinate axes at points A, B and C. The equations of the planes passing through A, B and C and parallel to the coordinate planes are, respectively,

$$x=a, y=b, z=c \tag{iii}$$

The locus of the point of intersection of these planes is obtained by eliminating the parameters a, b and c between (ii) and (iii). Putting the values of a, b and c from (iii) in (ii), the required locus is given by $\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=1$ or $\alpha x^{-1}+\beta y^{-1}+\gamma z^{-1}=1$

EXAMPLE 4.4

Show that the straight lines whose direction cosines are given by the equations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are parallel or perpendicular as

$$\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0 \text{ or } a^2(v+w)+b^2(w+u)+c^2(u+v)=0.$$

Sol. Here, $l=-\dfrac{(bm+cn)}{a}$ and $ul^2 + m^2v + wn^2 = 0$.

Eliminating l, we get

$$\frac{u(bm+cn)^2}{a^2}+vm^2+wn^2=0$$

$$u(b^2m^2+2bcmn+c^2n^2)+va^2m^2+wa^2n^2=0$$

$$(b^2u+a^2v)m^2+(2bcu)mn+(c^2u+a^2w)n^2=0$$

or

$$(b^2u+a^2v)\left(\frac{m}{n}\right)^2+(2bcu)\left(\frac{m}{n}\right)+(c^2u+a^2w)=0$$

which is quadratic in (m/n) having roots m_1/n_1 and m_2/n_2.

a. If the straight lines are parallel, the quadratic in m/n has equal roots, i.e., discriminant = 0

$$\Rightarrow \quad (2bcu)^2-4(b^2u+a^2v)(c^2u+a^2w)=0$$

or $$b^2c^2u^2=(b^2u+a^2v)(c^2u+a^2w)$$

or $$a^2vw+b^2uw+c^2uv=0$$

or $$\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$$

b. If the straight lines are perpendicular, then

$$\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{c^2u+a^2w}{b^2u+a^2v} \quad \text{(product of roots)}$$

$$\frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{b^2u + a^2v} \quad \text{(i)}$$

or

Similarly, by eliminating n, we get

$$\frac{l_1l_2}{b^2w + c^2v} = \frac{m_1m_2}{c^2u + a^2w} \quad \text{(ii)}$$

From (i) and (ii),

$$\frac{l_1l_2}{b^2w + c^2v} = \frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{b^2u + a^2v} = \lambda$$

Since they are perpendicular,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

or $$\lambda(b^2w + c^2v) + \lambda(c^2u + a^2w) + \lambda(b^2u + a^2v) = 0$$

or $$a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$$

EXAMPLE 4.5

Find the perpendicular distance of a corner of a cube of unit side length from a diagonal not passing through it.

Sol. Let the edges OA, OB and OC of the unit cube be along OX, OY and OZ, respectively.

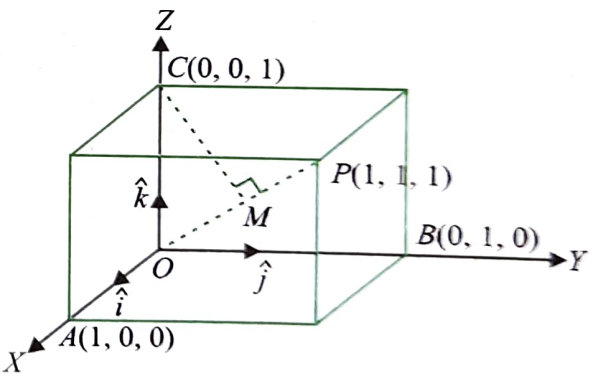
Since $OA = OB = OC = 1$ unit,

$$\overrightarrow{OA} = \hat{i},\ \overrightarrow{OB} = \hat{j} \text{ and } \overrightarrow{OC} = \hat{k}$$

Let CM be perpendicular from the corner C on the diagonal OP.

The vector equation of OP is

$$\vec{r} = \lambda(\hat{i} + \hat{j} + \hat{k})$$



$$OM = \text{projection of } \overrightarrow{OC} \text{ on } \overrightarrow{OP} = \frac{\overrightarrow{OC} \cdot \overrightarrow{OP}}{|\overrightarrow{OP}|}$$

$$= \hat{k} \cdot \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Now $$OC^2 = OM^2 + CM^2$$

or $$CM^2 = OC^2 - OM^2 = 1 - \frac{1}{3} = \frac{2}{3}$$

or $$CM = \sqrt{\frac{2}{3}}$$

EXAMPLE 4.6

A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the coordinate axes at A, B and C. If the planes through A, B and C parallel to the planes $x = 0$, $y = 0$ and $z = 0$, respectively, intersect at Q, find the locus of Q.

Sol. The given plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Let $P(h, k, l)$ be the point on the plane. Then

$$\frac{h}{a}+\frac{k}{b}+\frac{l}{c}=1$$

$\Rightarrow$ $$OP=\sqrt{h^2+k^2+l^2}$$

Direction cosines of OP are

$$\frac{h}{\sqrt{h^2+k^2+l^2}}, \frac{k}{\sqrt{h^2+k^2+l^2}} \text{ and } \frac{l}{\sqrt{h^2+k^2+l^2}}.$$

The equation of the plane through P and normal to OP is

$$\frac{hx}{\sqrt{h^2+k^2+l^2}}+\frac{ky}{\sqrt{h^2+k^2+l^2}}+\frac{lz}{\sqrt{h^2+k^2+l^2}}=\sqrt{h^2+k^2+l^2}$$

or $$hx+ky+lz=h^2+k^2+l^2$$

Therefore, $$A\equiv\left(\frac{h^2+k^2+l^2}{h}, 0, 0\right),$$

$$B\equiv\left(0, \frac{h^2+k^2+l^2}{k}, 0\right)$$

and $$C\equiv\left(0, 0, \frac{h^2+k^2+l^2}{l}\right)$$

If $Q(\alpha, \beta, \gamma)$, then

$$\alpha=\frac{h^2+k^2+l^2}{h}, \beta=\frac{h^2+k^2+l^2}{k} \text{ and } \gamma=\frac{h^2+k^2+l^2}{l}$$

Now, $$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{h^2+k^2+l^2}{\left(h^2+k^2+l^2\right)^2}=\frac{1}{h^2+k^2+l^2}$$

From (iii), $h=\frac{h^2+k^2+l^2}{\alpha}$ or $\frac{h}{a}=\frac{h^2+k^2+l^2}{a\alpha}$

Similarly, $\frac{k}{b}=\frac{h^2+k^2+l^2}{b\beta}$ and $\frac{l}{c}=\frac{h^2+k^2+l^2}{c\gamma}$

$$\frac{h^2+k^2+l^2}{a\alpha}+\frac{h^2+k^2+l^2}{b\beta}+\frac{h^2+k^2+l^2}{c\gamma}=\frac{h}{a}+\frac{k}{b}+\frac{l}{c}=1$$ [from (i)

or $$\frac{1}{a\alpha}+\frac{1}{b\beta}+\frac{1}{c\gamma}=\frac{1}{h^2+k^2+l^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$$ [from (iv)

The required equation of locus is

$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$$

EXAMPLE 4.7

If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

Sol. The given planes are

$$x - cy - bz = 0 \quad \text{(i)}$$

$$cx - y + az = 0 \quad \text{(ii)}$$

$$bx + ay - z = 0 \quad \text{(iii)}$$

The equation of the planes passing through the line of intersection of planes (i) and (ii) may be taken as

$$(x - cy - bz) + \lambda(cx - y + az) = 0$$

or $$x(1 + \lambda c) - y(c + \lambda) + z(-b + a\lambda) = 0 \quad \text{(iv)}$$

If planes (iii) and (iv) are the same, then Eqs. (iii) and (iv) will be identical.

$$\frac{1 + c\lambda}{b} = \frac{-(c + \lambda)}{a} = \frac{-b + a\lambda}{-1}$$

$$\lambda = -\frac{(a + bc)}{(ac + b)} \text{ and } \lambda = -\frac{(ab + c)}{(1 - a^2)}$$

$\therefore$ $$\frac{-(a + bc)}{(ac + b)} = -\frac{(ab + c)}{(1 - a^2)}$$

$$a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$$

or $$2a^2bc + ac^2 + ab^2 + a^3 - a = 0$$

or $$a(2abc + c^2 + b^2 + a^2 - 1) = 0$$

or $$a^2 + b^2 + c^2 + 2abc = 1$$

Alternate method:

Since the planes pass through the origin, the given planes have a common line of intersection if given system of equations has a non-trivial solution. Hence,

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

or $$a^2 + b^2 + c^2 + 2abc = 1$$

EXAMPLE 4.8

P is a point and PM and PN are the perpendiculars from P to z–x and x–y planes. If OP makes angles θ, α, β and γ with the plane OMN and the x–y, y–z and z–x planes, respectively, then prove that $\text{cosec}^2\,\theta = \text{cosec}^2\,\alpha + \text{cosec}^2\,\beta + \text{cosec}^2\,\gamma$.

Sol. Let P be (x_1, y_1, z_1). Point M is $(x_1, 0, z_1)$ and N is $(x_1, y_1, 0)$.

So normal to plane OMN is $\overrightarrow{OM} \times \overrightarrow{ON} = \vec{x}$ (say). Therefore,

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & 0 & z_1 \\ x_1 & y_1 & 0 \end{vmatrix} c = \hat{i}(-y_1 z_1) - \hat{j}(-x_1 z_1) + \hat{k}(x_1 y_1)$$

$$\sin\theta = \frac{-x_1 y_1 z + x_1 y_1 z + x_1 y_1 z_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{\Sigma x_1^2 y_1^2}}$$

$$\left(\text{because } \sin\theta = \frac{\vec{n} \times \vec{OP}}{|\vec{n}|\,|\vec{OP}|}\right)$$

$$\Rightarrow \quad \operatorname{cosec}^2\theta = \frac{\Sigma x_1^2 \Sigma x_1^2 y_1^2}{(x_1 y_1 z_1)^2} = \frac{\Sigma x_1^2}{x_1^2} + \frac{\Sigma x_1^2}{y_1^2} + \frac{\Sigma x_1^2}{z_1^2}$$

Now, $\sin\alpha = \dfrac{\overrightarrow{OP}\cdot\hat{k}}{|\overrightarrow{OP}|} = \dfrac{z_1}{\sqrt{\Sigma x_1^2}}$, $\sin\beta = \dfrac{x_1}{\sqrt{\Sigma x_1^2}}$ and

$$\sin\gamma = \frac{y_1}{\sqrt{\Sigma x_1^2}}$$

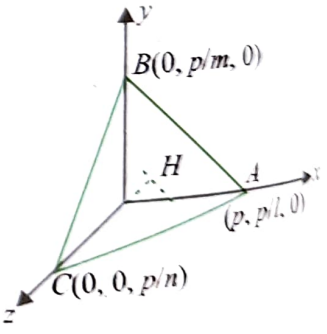
Now, $\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma = \dfrac{\Sigma x_1^2}{x_1^2} + \dfrac{\Sigma x_1^2}{y_1^2} + \dfrac{\Sigma x_1^2}{z_1^2} = \operatorname{cosec}^2\theta$

Hence proved.

EXAMPLE 4.9

A variable plane $lx + my + nz = p$ (where l, m, n are direction cosines of normal) intersects the coordinate axes at points A, B and C, respectively. Show that the foot of the normal on the plane from the origin is the orthocentre of triangle ABC and hence find the coordinates of the circumcentre of triangle ABC.

Sol. $\dfrac{x}{p/l} + \dfrac{y}{p/m} + \dfrac{z}{p/n} = 1$



The foot of normal on plane has coordinates $H\,(lp, mp, np)$.

Direction ratios of AH are $lp - (p/l)$, mp and np and direction ratios of BC are 0, $-p/m$ and p/n. Thus,

$$\left(lp - \frac{p}{l}\right)\cdot 0 + (mp)\left(-\frac{p}{m}\right) + (np)\left(\frac{p}{n}\right) = 0$$

Hence, AH is perpendicular to BC.

Similarly, BH is perpendicular to AC and CH is perpendicular to AB.

Hence, H is the orthocenter.

Moreover, in any triangle, G (centroid) divides OH in the ratio 1 : 2. Hence,

$$G \equiv \left(\frac{p}{3l}, \frac{p}{3m}, \frac{p}{3n}\right)$$

$$H \equiv (lp, mp, np)$$

$$\Rightarrow \quad O \equiv \left(\frac{p - l^2 p}{2l}, \frac{p - m^2 p}{2m}, \frac{p - n^2 p}{2n}\right).$$

EXAMPLE 4.10

Let $x-y\sin\alpha-z\sin\beta=0$, $x\sin\alpha+z\sin\gamma-y=0$ and $x\sin\beta+y\sin\gamma-z=0$ be the equations of the planes such that $\alpha+\beta+\gamma=\pi/2$ (where α, β and $\gamma\neq 0$). Then show that there is a common line of intersection of the three given planes.

Sol.

$$x-y\sin\alpha-z\sin\beta=0 \quad \text{(i)}$$

$$x\sin\alpha+z\sin\gamma-y=0 \quad \text{(ii)}$$

$$x\sin\beta+y\sin\gamma-z=0 \quad \text{(iii)}$$

These planes pass through the origin. Let l, m and n be the direction cosines of the line of intersection of planes (i) and (ii). Then

$$l\cdot 1-m\sin\alpha-n\sin\beta=0$$

$$l\sin\alpha-m\cdot 1+n\sin\gamma=0$$

$$\Rightarrow \quad \frac{l}{-\sin\gamma\sin\alpha-\sin\beta}=\frac{m}{-\sin\beta\sin\alpha-\sin\gamma}=\frac{n}{-1+\sin^2\alpha} \quad \text{(iv)}$$

If $\alpha+\beta+\gamma=\dfrac{\pi}{2}\Rightarrow\beta=\dfrac{\pi}{2}-(\alpha+\gamma)$

$$\sin\beta=\sin\left(\frac{\pi}{2}-(\alpha+\gamma)\right)=\cos(\alpha+\gamma)$$

$$\sin\beta=\cos\alpha\cos\gamma-\sin\alpha\sin\gamma$$

$$\sin\beta+\sin\alpha\sin\gamma=\cos\alpha\cos\gamma$$

Similarly, $\sin\gamma+\sin\beta\sin\alpha=\cos\alpha\cos\beta$

From equation (iv), we get

$$\frac{l}{\cos\alpha\cos\gamma}=\frac{m}{\cos\alpha\cos\beta}=\frac{n}{\cos^2\alpha}$$

$$\frac{l}{\cos\gamma}=\frac{m}{\cos\beta}=\frac{n}{\cos\alpha} \quad \text{(v)}$$

The line of intersection of planes (i) and (ii) also passes through the origin. Then the equation of the line is

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n}$$

$$\Rightarrow \quad \frac{x}{\cos\gamma}=\frac{y}{\cos\beta}=\frac{z}{\cos\alpha} \quad \text{(vi)}$$

If the line also lies on plane (iii), then the three planes will intersect on this straight line.

The angle between line and normal of plane (iii) should be $\pi/2$. Therefore,

$$\cos\gamma\sin\beta+\cos\beta\sin\gamma+\cos\alpha(-1)=\sin(\beta+\gamma)-\cos\alpha$$

$$=\sin\left(\frac{\pi}{2}-\alpha\right)-\cos\alpha=0$$

Hence, $\dfrac{x}{\cos\gamma}=\dfrac{y}{\cos\beta}=\dfrac{z}{\cos\alpha}$ is the common line of the intersection of the three given planes.

EXAMPLE 4.11

Let a plane $ax + by + cz + 1 = 0$, where a, b and c are parameters, make an angle $60°$ with the line $x = y = z$, $45°$ with the line $x = y - z = 0$ and θ with the plane $x = 0$. The distance of the plane from point (2, 1, 1) is 3 units. Find the value of θ and the equation of the plane.

Sol. $ax + by + cz + 1 = 0$

It makes an angle $60°$ with the line $x = y = z$. So we get

$$\sin 60° = \frac{a + b + c}{\sqrt{3}\,\Sigma a^2}$$

or $$3\sqrt{\Sigma a^2} = 2(a + b + c)$$

Plane (i) makes an angle of $45°$ with the line

$$x = y - z = 0 \left(\text{or } \frac{x}{0} = \frac{y}{1} = \frac{z}{1}\right)$$

$$\sin 45° = \frac{b + c}{\sqrt{2}\sqrt{\Sigma a^2}} \text{ or } \sqrt{\Sigma a^2} = b + c$$

Plane (i) makes an angle θ with the plane $x = 0$. So we get

$$\cos\theta = \frac{a}{\sqrt{\Sigma a^2}}$$

From (ii) and (iii), we get

$$(\sqrt{\Sigma a^2}) = 2a$$

or $$\frac{a}{\sqrt{\Sigma a^2}} = \frac{1}{2}$$

From (iv), $\cos\theta = 1/2$ or $\theta = 60°$

Distance of plane (i) from the point (2, 1, 1) is 3 units.

$\Rightarrow$ $$\frac{2a + b + c + 1}{\sqrt{\Sigma a^2}} = \pm 3$$

or $$\pm 3\sqrt{\Sigma a^2} = 2a + b + c + 1$$

Case I:

$$3\sqrt{\Sigma a^2} = 2a + b + c + 1$$

From (ii) and (v), we get

$$b + c - 1 = 0$$

and from (iii) and (iv), we get

$$2a + b + c + 1 = 3(b + c)$$

From (vi) and (vii), we get

$$a = \frac{1}{2},\ b = \frac{(2 \mp \sqrt{2})}{4} \text{ and } c = \frac{2 \pm \sqrt{2}}{4}.$$

Hence, the set of such planes is

$$2x + (2 \pm \sqrt{2})y + (2 \pm \sqrt{2})z + 4 = 0.$$

Case II:

$$-3\sqrt{\Sigma a^2} = 2a + b + c + 1$$

$$a = \frac{-1}{10},\ b = \frac{-(2 \pm \sqrt{2})}{20} \text{ and } c = \frac{-(2 \mp \sqrt{2})}{20}$$

Hence, the other set of the planes is

$$2x + (2 \pm \sqrt{2})y + (2 \mp \sqrt{2})z - 20 = 0.$$

EXAMPLE 4.12

Prove that for all values of λ and μ, the planes $\frac{2x}{a} + \frac{y}{b} + \frac{2z}{c} - 1 + \lambda\left(\frac{x}{a} - \frac{2y}{b} - \frac{z}{c} - 2\right) = 0$ and $\frac{4x}{a} - \frac{3y}{b} - 5 + \mu\left(\frac{5y}{b} + \frac{4z}{c} + 3\right) = 0$ intersect on the same line.

Sol. Let the given planes intersect on the line with direction ratios l, m and n. In that case,

$$(2 + \lambda)\frac{l}{a} + (1 - 2\lambda)\frac{m}{b} + (2 - \lambda)\frac{n}{c} = 0 \qquad \text{(i)}$$

and

$$\frac{4l}{a} - (3 - 5\mu)\frac{m}{b} + 4\mu.\frac{n}{c} = 0 \qquad \text{(ii)}$$

Hence,

$$\frac{l/a}{6 - 6\mu - 3\lambda - 3\lambda\mu} = \frac{m/b}{8 - 8\mu - 4\lambda - 4\lambda\mu} = \frac{n/c}{-10 + 10\mu + 5\lambda + 5\lambda\mu}$$

or

$$\frac{l/a}{3(2 - 2\mu - \lambda - \lambda\mu)} = \frac{m/b}{4(2 - 2\mu - \lambda - \lambda\mu)} = \frac{n/c}{-5(2 - 2\mu - \lambda - \lambda\mu)}$$

or

$$\frac{l/a}{3} = \frac{m/b}{4} = \frac{n/c}{-5} \qquad (\text{provided } 2 - 2\mu - \lambda - \lambda\mu \neq 0)$$

which are independent of λ and μ. Hence, a line with direction ratios $(3a, 4b, -5c)$ lies in both the planes.

For $2 - 2\mu - \lambda - \lambda\mu = 0$ or $\lambda = \frac{2(1 - \mu)}{1 + \mu}$, planes (i) and (ii) coincide with each other. Hence, the two given families of planes intersect on the same line.

EXAMPLE 4.13

OA, OB and OC, with O as the origin, are three mutually perpendicular lines whose direction cosines are l_r, m_r and n_r $(r = 1, 2 \text{ and } 3)$. If the projections of OA and OB on the plane $z = 0$ make angles ϕ_1 and ϕ_2, respectively, with the x-axis, prove that $\tan(\phi_1 - \phi_2) = \pm n_3/n_1 n_2$.

Sol. Let A_1 and B_1 be the projections of A and B on the plane $z = 0$. Let OA, OB and OC be of the unit length each so that the coordinates of A, B and C are $A(l_1, m_1, n_1)$, $B(l_2, m_2, n_2)$ and $C(l_3, m_3, n_3)$. The coordinates of A_1 and B_1, therefore, are $A_1(l_1, m_1, 0)$ and $B_1(l_2, m_2, 0)$. Since OA_1 and OB_1 make angles ϕ_1 and ϕ_2, respectively, with the x-axis, the angle between OA_1 and OB_1 is $\phi_1 \sim \phi_2$. Hence,

$$\cos(\phi_1 - \phi_2) = \frac{l_1 l_2 + m_1 m_2}{\sqrt{l_1^2 + m_1^2}\sqrt{l_2^2 + m_2^2}} \quad \text{(i)}$$

Also OA, OB and OC are mutually perpendicular so that

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

and

$$l_1^2 + m_1^2 + n_1^2 = 1$$

Eq. (i), therefore, yields

$$\cos(\phi_1 - \phi_2) = \frac{-n_1 n_2}{\sqrt{1 - n_1^2}\sqrt{1 - n_2^2}}$$

or

$$\sec^2(\phi_1 - \phi_2) = \frac{1 - n_1^2 - n_2^2 + n_1^2 n_2^2}{n_1^2 n_2^2}$$

$$= 1 + \frac{1 - n_1^2 - n_2^2}{n_1^2 n_2^2} = 1 + \frac{n_3^2}{n_1^2 n_2^2}$$

or

$$\tan^2(\phi_1 - \phi_2) = \frac{n_3^2}{n_1^2 n_2^2}$$

or

$$\tan(\phi_1 - \phi_2) = \pm\frac{n_3}{n_1 n_2}$$

EXAMPLE 4.14

O is the origin and lines OA, OB and OC have direction cosines l_r, m_r and n_r ($r = 1, 2$ and 3). If lines OA', OB' and OC' bisect angles BOC, COA and AOB, respectively, prove that planes AOA', BOB' and COC' pass through the line $\dfrac{x}{l_1 + l_2 + l_3} = \dfrac{y}{m_1 + m_2 + m_3} = \dfrac{z}{n_1 + n_2 + n_3}$.

Sol. If θ is the angle BCO, then the direction cosines of OA' (bisector of $\angle BOC$) are $\dfrac{l_2 + l_3}{2\cos(\theta/2)}$, $\dfrac{m_2 + m_3}{2\cos(\theta/2)}$ and $\dfrac{n_2 + n_3}{2\cos(\theta/2)}$ or the direction ratios of OA' are $l_2 + l_3$, $m_2 + m_3$ and $n_2 + n_3$.

Also, the direction cosines of OA are l_1, m_1 and n_1. Hence, the equation of plane AOA' is

$$\begin{vmatrix} x & y & z \\ l_2 + l_3 & m_2 + m_3 & n_2 + n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 + R_3$, we get the equation of plane AOA' as

$$\begin{vmatrix} x & y & z \\ l_1 + l_2 + l_3 & m_1 + m_2 + m_3 & n_1 + n_2 + n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

For all values of r, the point $((l_1 + l_2 + l_3)r, (m_1 + m_2 + m_3)r$ and $(n_1 + n_2 + n_3)r)$ lies on plane AOA'. Hence, the line $\frac{x}{l_1+l_2+l_3} = \frac{y}{m_1+m_2+m_3} = \frac{z}{n_1+n_2+n_3} = r$ lies on plane AOA'. Similarly, this line lies on planes BOB' and COC' also. Hence, all the three planes, AOA', BOB' and COC', pass through the line.

EXAMPLE 4.15

If P is any point on the plane $lx + my + nz = p$ and Q is a point on the line OP such that $OP \cdot OQ = p^2$, then find the locus of the point Q.

Sol. Let $P(\alpha, \beta, \gamma)$ and $Q(x_1, y_1, z_1)$ be the given points.

Direction ratios of OP are α, β and γ and those of OQ are x_1, y_1 and z_1.

Since O, Q and P are collinear, we have

$$\frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1} = k \quad \text{(say)} \tag{i}$$

As $P(\alpha, \beta, \gamma)$ lies on the plane $lx + my + nz = p$,

$$l\alpha + m\beta + n\gamma = p$$

or $$klx_1 + kmy_1 + knz_1 = p \quad \text{[using (i)]} \tag{ii}$$

Since $OP \cdot OQ = p^2$, we have

$$\sqrt{\alpha^2 + \beta^2 + \gamma^2} \cdot \sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$$

or $$\sqrt{k^2x_1^2 + k^2y_1^2 + k^2z_1^2} \cdot \sqrt{x_1^2 + y_1^2 + z_1^2} = p^2$$

or $$k(x_1^2 + y_1^2 + z_1^2) = p^2 \tag{iii}$$

From (ii) and (iii),

$$\frac{lx_1 + my_1 + nz_1}{x_1^2 + y_1^2 + z_1^2} = \frac{1}{p}$$

or $$p(lx_1 + my_1 + nz_1) = (x_1^2 + y_1^2 + z_1^2)$$

Hence, the locus of Q is $p(lx + my + nz) = (x^2 + y^2 + z^2)$

EXAMPLE 4.16

If a variable plane forms a tetrahedron of constant volume $64k^3$ with the coordinate planes, find the locus of the centroid of the tetrahedron.

Sol. Let the variable plane intersect the coordinate axes at $A(a, 0, b)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Then, the equation of the plane will be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{i}$$

Let $P(\alpha, \beta, \gamma)$ be the centroid of tetrahedron $OABC$. Then,

or $$\alpha = \frac{a}{4}, \beta = \frac{b}{4} \text{ and } \gamma = \frac{c}{4}$$

$$a = 4\alpha, b = 4\beta \text{ and } c = 4\gamma$$

$\Rightarrow$ Volume of tetrahedron = (Area of ΔAOB) OC

or $$64k^3 = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{abc}{6}$$

$$= \frac{(4\alpha)(4\beta)(4\gamma)}{6} \text{ or } k^3 = \frac{\alpha\beta\gamma}{6}$$

Therefore, the required locus of $P(\alpha,\beta,\gamma)$ is $xyz = 6k^3$

EXAMPLE 4.17

Prove that the volume of tetrahedron bounded by the planes $\vec{r}\cdot(m\hat{j}+n\hat{k})=0$, $\vec{r}\cdot(n\hat{k}+\ell\hat{i})=0$, $\vec{r}\cdot(\ell\hat{i}+m\hat{j})=0$, $\vec{r}\cdot(\ell\hat{i}+m\hat{j}+n\hat{k})=p$ is $\frac{2p^3}{3\ell mn}$.

Sol. Planes $\vec{r}\cdot(m\hat{j}+n\hat{k})=0$ (i)

$\vec{r}\cdot(n\hat{k}+\ell\hat{i})=0$ (ii)

$\vec{r}\cdot(\ell\hat{i}+m\hat{j})=0$ (iii)

are passing through the origin O.

Let the planes (i) and (ii), (ii) and (iii), (i) and (iii) be intersecting along edges OA, OB and OC, where A, B, C are points where these vectors are tereminated by the plane

$\vec{r}\cdot(\ell\hat{i}+m\hat{j}+n\hat{k})=p$ (iv)

Vectors along the edges OA, OB, OC are, respectively,

$$(m\hat{j}+n\hat{k})\times(\ell\hat{i}+n\hat{k}) = mn\hat{i}+n\ell\hat{j}-\ell m\hat{k}$$

$$(n\hat{k}+\ell\hat{i})\times(\ell\hat{i}+m\hat{j}) = -mn\hat{i}+n\ell\hat{j}+\ell m\hat{k}$$

$$(\ell\hat{i}+m\hat{j})\times(m\hat{j}+n\hat{k}) = mn\hat{i}-n\ell\hat{j}+\ell m\hat{k}$$

Equation of line OA is $\vec{r}=\lambda(mn\hat{i}+n\ell\hat{j}-\ell m\hat{k})$

For its point of intersection with the plane $\vec{r}\cdot(\ell\hat{i}+m\hat{j}+n\hat{k})=p$, we have

$$\lambda(mn\hat{i}+n\ell\hat{j}-\ell m\hat{k})\cdot(\ell\hat{i}+m\hat{j}+n\hat{k}) = p$$

$\therefore$ $$\lambda = \frac{p}{\ell mn}$$

$\therefore$ $$\overrightarrow{OA} = \frac{p}{\ell mn}(mn\hat{i}+n\ell\hat{j}-\ell m\hat{k})$$

Similarly, $\overrightarrow{OB} = \frac{p}{\ell mn}(-mn\hat{i}+n\ell\hat{j}+\ell m\hat{k})$ and $\overrightarrow{OC} = \frac{p}{\ell mn}(mn\hat{i}-n\ell\hat{j}+\ell m\hat{k})$

Therefore, volume of tetrahedron by vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ is

$$\frac{1}{6}\left(\frac{p}{\ell mn}\right)^3\begin{vmatrix} mn & n\ell & -\ell m \\ -mn & n\ell & \ell m \\ mn & -n\ell & \ell m \end{vmatrix} = \frac{1}{6}\left(\frac{p}{\ell mn}\right)^3 \ell^2m^2n^2\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \frac{2}{3}\frac{p^3}{\ell mn}$$

EXAMPLE 4.18

A variable plane passes through a fixed point (a, b, c) and cuts the coordinate axes at points A, B and C. Show that the locus of the centre of the sphere $OABC$ is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$.

Sol. Let (α, β, γ) be any point on the locus. Then according to the given condition, (α, β, γ) is the centre of the sphere through the origin. Therefore, its equation is given by

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = (0-\alpha)^2 + (0-\beta)^2 + (0-\gamma)^2$$

$$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z = 0$$

To obtain its point of intersection with the x-axis, we put $y = 0$ and $z = 0$, so that

$$x^2 - 2\alpha x = 0 \text{ or } x(x - 2\alpha) = 0 \text{ or } x = 0, 2\alpha$$

Thus, the plane meets the x-axis at $O(0, 0, 0)$ and $A(2\alpha, 0, 0)$. Similarly, it meets the y-axis at $O(0, 0, 0)$ and $B(0, 2\beta, 0)$, and the z-axis at $O(0, 0, 0)$ and $C(0, 0, 2\gamma)$.

The equation of the plane through A, B and C is

$$\frac{x}{2\alpha} + \frac{y}{2\beta} + \frac{z}{2\gamma} = 1 \qquad \text{(intercept form)}$$

Since it passes through (a, b, c), we get

$$\frac{a}{2\alpha} + \frac{b}{2\beta} + \frac{c}{2\gamma} = 1 \text{ or } \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2$$

Hence, locus of (α, β, γ) is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

Exercises

Single Correct Answer Type

1. In a three-dimensional xyz space, the equation $x^2 - 5x + 6 = 0$ represents

(1) points (2) planes (3) curves (4) pair of straight lines

2. The line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1}$ intersects the curve $xy = c^2, z = 0$ if c is equal to

(1) ± 1 (2) $\pm 1/3$ (3) $\pm \sqrt{5}$ (4) none of these

3. Let the equations of a line and a plane be $\frac{x+3}{2} = \frac{y-4}{3} = \frac{z+5}{2}$ and $4x - 2y - z = 1$, respectively, then

(1) the line is parallel to the plane
(2) the line is perpendicular to the plane
(3) the line lies in the plane
(4) none of these

4. The length of the perpendicular from the origin to the plane passing through the point $\vec{a}$ and containing the line $\vec{r} = \vec{b} + \lambda\vec{c}$ is

(1) $\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}|}$
(2) $\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{a}\times\vec{b} + \vec{b}\times\vec{c}|}$
(3) $\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{b}\times\vec{c} + \vec{c}\times\vec{a}|}$
(4) $\frac{[\vec{a}\ \vec{b}\ \vec{c}]}{|\vec{c}\times\vec{a} + \vec{a}\times\vec{b}|}$

5. The distance of point A (–2, 3, 1) from the line PQ through P (–3, 5, 2), which makes equal angles with the axes is

(1) $2/\sqrt{3}$ (2) $\sqrt{14/3}$ (3) $16/\sqrt{3}$ (4) $5/\sqrt{3}$

6. The Cartesian equation of the plane $\vec{r} = (1+\lambda-\mu)\hat{i} + (2-\lambda)\hat{j} + (3-2\lambda+2\mu)\hat{k}$ is

(1) $2x + y = 5$ (2) $2x - y = 5$ (3) $2x + z = 5$ (4) $2x - z = 5$

7. A unit vector parallel to the intersection of the planes $\vec{r}\cdot\left(\hat{i}-\hat{j}+\hat{k}\right) = 5$ and $\vec{r}\cdot\left(2\hat{i}+\hat{j}-3\hat{k}\right) = 4$ is

(1) $\frac{2\hat{i}+5\hat{j}-3\hat{k}}{\sqrt{38}}$ (2) $\frac{2\hat{i}-5\hat{j}+3\hat{k}}{\sqrt{38}}$ (3) $\frac{-2\hat{i}-5\hat{j}-3\hat{k}}{\sqrt{38}}$ (4) $\frac{-2\hat{i}+5\hat{j}-3\hat{k}}{\sqrt{38}}$

8. Let L_1 be the line $\vec{r_1} = 2\hat{i} + \hat{j} - \hat{k} + \lambda(\hat{i} + 2\hat{k})$ and let L_2 be the line $\vec{r_2} = 3\hat{i} + \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k})$. Let π be the plane which contains the line L_1 and is parallel to L_2. The distance of the plane π from the origin is

(1) $\sqrt{2/7}$ (2) 1/7 (3) $\sqrt{6}$ (4) none of these

9. For the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$, which one of the following is incorrect?

(1) It lies in the plane $x - 2y + z = 0$
(2) It is same as line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$
(3) It passes through (2, 3, 5)
(4) It is parallel to the plane $x - 2y + z - 6 = 0$

10. The value of m for which straight line $3x - 2y + z + 3 = 0 = 4x - 3y + 4z + 1$ is parallel to the plane $2x - y + mz - 2 = 0$ is

(1) -2 (2) 8 (3) -18 (4) 11

11. The intercept made by the plane $\vec{r} \cdot \vec{n} = q$ on the x-axis is

(1) $\frac{q}{\hat{i} \cdot \vec{n}}$ (2) $\frac{\hat{i} \cdot \vec{n}}{q}$ (3) $\frac{\hat{i} \cdot \vec{n}}{q}$ (4) $\frac{q}{|\vec{n}|}$

12. Equation of a line in the plane $\pi \equiv 2x - y + z - 4 = 0$ which is perpendicular to the line l whose equation is $\frac{x-2}{1} = \frac{y-2}{-1} = \frac{z-3}{-2}$ and which passes through the point of intersection of l and π is

(1) $\frac{x-2}{1} = \frac{y-1}{5} = \frac{z-1}{-1}$ (2) $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-5}{-1}$

(3) $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z+1}{1}$ (4) $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-1}{1}$

13. If the foot of the perpendicular from the origin to a plane is $P(a, b, c)$, the equation of the plane is

(1) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$ (2) $ax + by + cz = 3$

(3) $ax + by + cz = a^2 + b^2 + c^2$ (4) $ax + by + cz = a + b + c$

14. The equation of a plane which passes through the point of intersection of lines $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$, and $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and at greatest distance from point (0, 0, 0) is

(1) $4x + 3y + 5z = 25$ (2) $4x + 3y = 5z = 50$ (3) $3x + 4y + 5z = 49$ (4) $x + 7y - 5z = 2$

15. Let $A(\vec{a})$ and $B(\vec{b})$ be points on two skew lines $\vec{r} = \vec{a} + \lambda \vec{p}$ and $\vec{r} = \vec{b} + u\vec{q}$ and the shortest distance between the skew lines is 1, where $\vec{p}$ and $\vec{q}$ are unit vectors forming adjacent sides of a parallelogram enclosing an area of $\frac{1}{2}$ units. If an angle between AB and the line of shortest distance is $60°$, then $AB =$

(1) $\frac{1}{2}$ (2) 2 (3) 1 (4) $\lambda \in R - \{0\}$

16. Let $A(1, 1, 1)$, $B(2, 3, 5)$ and $C(-1, 0, 2)$ be three points, then equation of a plane parallel to the plane ABC which is at distance 2 is

(1) $2x - 3y + z + 2\sqrt{14} = 0$ (2) $2x - 3y + z - \sqrt{14} = 0$

(3) $2x - 3y + z + 2 = 0$ (4) $2x - 3y + z - 2 = 0$

17. The point on the line $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+5}{-2}$ at a distance of 6 from the point $(2, -3, -5)$ is

(1) $(3, -5, -3)$ (2) $(4, -7, -9)$ (3) $(0, 2, -1)$ (4) $(-3, 5, 3)$

18. The coordinates of the foot of the perpendicular drawn from the origin to the line joining the points $(-9, 4, 5)$ and $(10, 0, -1)$ will be

(1) $(-3, 2, 1)$ (2) $(1, 2, 2)$ (3) $(4, 5, 3)$ (4) none of these

19. If $P_1: \vec{r}\cdot\vec{n}_1 - d_1 = 0, P_2: \vec{r}\cdot\vec{n}_2 - d_2 = 0$ and $P_3: \vec{r}\cdot\vec{n}_3 - d_3 = 0$ are three planes and $\vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ are three non-coplanar vectors, then three lines $P_1 = 0, P_2 = 0$; $P_2 = 0, P_3 = 0$ and $P_3 = 0, P_1 = 0$ are

(1) parallel lines (2) coplanar lines (3) coincident lines (4) concurrent lines

20. The length of projection of the line segment joining the points $(1, 0, -1)$ and $(-1, 2, 2)$ on the plane $x + 3y - 5z = 6$ is equal to

(1) 2 (2) $\sqrt{\frac{271}{53}}$ (3) $\sqrt{\frac{472}{31}}$ (4) $\sqrt{\frac{474}{35}}$

21. The number of planes that are equidistant from four non-coplanar points is

(1) 3 (2) 4 (3) 7 (4) 9

22. In a three-dimensional coordinate system, P, Q and R are images of a point $A\ (a, b, c)$ in the x–y, y–z and z–x planes, respectively. If G is the centroid of triangle PQR, then area of triangle AOG is (O is the origin)

(1) 0 (2) $a^2 + b^2 + c^2$ (3) $\frac{2}{3}(a^2 + b^2 + c^2)$ (4) none of these

23. A plane passing through (1, 1, 1) cuts positive direction of coordinate axes at A, B and C, then the volume of tetrahedron $OABC$ satisfies

(1) $V \le \frac{9}{2}$ (2) $V \ge \frac{9}{2}$ (3) $V = \frac{9}{2}$ (4) none of these

24. If lines $x = y = z$ and $x = \frac{y}{2} = \frac{z}{3}$ and third line passing through (1, 1, 1) form a triangle of area $\sqrt{6}$ units, then the point of intersection of third line with the second line will be

(1) (1, 2, 3) (2) (2, 4, 6) (3) $\left(\frac{4}{3}, \frac{8}{3}, \frac{12}{3}\right)$ (4) none of these

25. The point of intersection of the line passing through (0, 0, 1) and intersecting the lines $x + 2y + z = 1$, $-x + y - 2z = 2$ and $x + y = 2, x + z = 2$ with xy plane is

(1) $\left(\frac{5}{3}, -\frac{1}{3}, 0\right)$ (2) (1, 1, 0) (3) $\left(\frac{2}{3}, -\frac{1}{3}, 0\right)$ (4) $\left(-\frac{5}{3}, \frac{1}{3}, 0\right)$

26. Shortest distance between the lines $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1}$ and $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{1}$ is equal to

(1) $\sqrt{14}$ (2) $\sqrt{7}$ (3) $\sqrt{2}$ (4) none of these

27. Distance of point $P\ (\vec{p})$ from the plane $\vec{r}\cdot\vec{n} = 0$ is

(1) $|\vec{p}\cdot\vec{n}|$ (2) $\frac{|\vec{p}\times\vec{n}|}{|\vec{n}|}$ (3) $\frac{|\vec{p}\cdot\vec{n}|}{|\vec{n}|}$ (4) none of these

28. The reflection of the point $\vec{a}$ in the plane $\vec{r}\cdot\vec{n} = q$ is

(1) $\vec{a} + \frac{(\vec{q} - \vec{a}\cdot\vec{n})}{|\vec{n}|}$ (2) $\vec{a} + 2\left(\frac{(\vec{q} - \vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}$

(3) $\vec{a} + \frac{2(\vec{q} + \vec{a}\cdot\vec{n})}{|\vec{n}|}\vec{n}$ (4) none of these

29. Line $\vec{r} = \vec{a} + \lambda \vec{b}$ will not meet the plane $\vec{r} \cdot \vec{n} = q$, if

(1) $\vec{b} \cdot \vec{n} = 0, \vec{a} \cdot \vec{n} = q$ (2) $\vec{b} \cdot \vec{n} \neq 0, \vec{a} \cdot \vec{n} \neq q$ (3) $\vec{b} \cdot \vec{n} = 0, \vec{a} \cdot \vec{n} \neq q$ (4) $\vec{b} \cdot \vec{n} \neq 0, \vec{a} \cdot \vec{n} = q$

30. If a line makes an angle of $\frac{\pi}{4}$ with the positive direction of each of x-axis and y-axis, then the angle that the line makes with the positive direction of the z-axis is

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{6}$

31. The ratio in which the plane $\vec{r} \cdot (\vec{i} - 2\vec{j} + 3\vec{k}) = 17$ divides the line joining the points $-2\vec{i} + 4\vec{j} + 7\vec{k}$ and $3\vec{i} - 5\vec{j} + 8\vec{k}$ is

(1) 1 : 5 (2) 1 : 10 (3) 3 : 5 (4) 3 :10

32. The image of the point (–1, 3, 4) in the plane $x - 2y = 0$ is

(1) $\left(-\frac{17}{3}, -\frac{19}{3}, 4\right)$ (2) (15, 11, 4) (3) $\left(-\frac{17}{3}, -\frac{19}{3}, 1\right)$ (4) $\left(\frac{9}{5}, -\frac{13}{5}, 4\right)$

33. The distance between the line $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$ is

(1) $\frac{10}{3\sqrt{3}}$ (2) $\frac{10}{9}$ (3) $\frac{10}{3}$ (4) $\frac{3}{10}$

34. Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$. If L makes an angle α with the positive x-axis, then $\cos \alpha$ equals

(1) $\frac{1}{2}$ (2) 1 (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{\sqrt{3}}$

35. The length of the perpendicular drawn from (1, 2, 3) to the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is

(1) 4 (2) 5 (3) 6 (4) 7

36. If angle θ between the line $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda}\, z + 4 = 0$ is such that $\sin\theta = \frac{1}{3}$, the value of λ is

(1) $\frac{-3}{5}$ (2) $\frac{5}{3}$ (3) $\frac{-4}{3}$ (4) $\frac{3}{4}$

37. The intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$ is the same as the intersection of one of the spheres and the plane

(1) $x - y - z = 1$ (2) $x - 2y - z = 1$ (3) $x - y - 2z = 1$ (4) $2x - y - z = 1$

38. A plane makes intercepts OA, OB and OC whose measurements are b and c on the OX, OY and OZ axes. The area of triangle ABC is

(1) $\frac{1}{2}(ab + bc + ca)$ (2) $\frac{1}{2}abc\,(a + b + c)$

(3) $\frac{1}{2}(a^2b^2 + b^2c^2 + c^2a^2)^{1/2}$ (4) $\frac{1}{2}(a + b + c)^2$

39. A line makes an angle θ with each of the x- and z-axes. If the angle β, which it makes with the y-axis, is such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals

(1) $\frac{2}{3}$ (2) $\frac{1}{5}$ (3) $\frac{3}{5}$ (4) $\frac{2}{5}$

40. The shortest distance from the plane $12x + y + 3z = 327$ to the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 155$ is

(1) 39 (2) 26 (3) $41\frac{4}{13}$ (4) 13

41. A tetrahedron has vertices $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$, then angle between faces OAB and ABC will be

(1) $\cos^{-1}\left(\frac{17}{31}\right)$ (2) 30° (3) 90° (4) $\cos^{-1}\left(\frac{19}{35}\right)$

42. The radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2z - 2y - 4z - 19 = 0$ is cut by the plane $x + 2y + 2z + 7 = 0$ is

(1) 2 (2) 3 (3) 4 (4) 1

43. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if

(1) $k = 1$ or -1 (2) $k = 0$ or -3 (3) $k = 3$ or -3 (4) $k = 0$ or -1

44. The point of intersection of the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-36} = \frac{y-3}{2} = \frac{z-6}{4}$ is

(1) $\left(21, \frac{5}{3}, \frac{10}{3}\right)$ (2) (2, 10, 4) (3) (–3, 3, 6) (4) (5, 7, –2)

45. Two systems of rectangular axes have the same origin. If a plane cuts them at distance a, b, c and a', b', c' from the origin, then

(1) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}=0$ (2) $\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}+\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}=0$

(3) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}=0$ (4) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}=0$

46. The plane which passes through the point (3, 2, 0) and the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$ is

(1) $x - y + z = 1$ (2) $x + y + z = 5$ (3) $x + 2y - z = 1$ (4) $2x - y + z = 5$

47. The direction ratios of a normal to the plane through (1, 0, 0) and (0, 1, 0), which makes an angle of $\frac{\pi}{4}$ with the plane $x + y = 3$, are

(1) $<1, \sqrt{2}, 1>$ (2) $<1, 1, \sqrt{2}>$ (3) $<1, 1, 2>$ (4) $<\sqrt{2}, 1, 1>$

48. The centre of the circle given by $\vec{r}\cdot(\hat{i}+2\hat{j}+2\hat{k}) = 15$ and $|\vec{r}-(\hat{j}+2\hat{k})| = 4$ is

(1) (0, 1, 2) (2) (1, 3, 4) (3) (– 1, 3, 4) (4) none of these

49. The lines which intersect the skew lines $y = mx, z = c$; $y = -mx, z = -c$ and the x-axis lie on the surface

(1) $cz = mxy$ (2) $xy = cmz$ (3) $cy = mxz$ (4) none of these

50. Distance of the point $P(\vec{p})$ from the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is

(1) $\left|(\vec{a}-\vec{p}) + \frac{((\vec{p}-\vec{a})\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}\right|$ (2) $\left|(\vec{b}-\vec{p}) + \frac{((\vec{p}-\vec{a})\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}\right|$

(3) $\left|(\vec{a}-\vec{p}) + \frac{((\vec{p}-\vec{b})\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}\right|$ (4) none of these

51. From the point $P(a, b, c)$, let perpendiculars PL and PM be drawn to YOZ and ZOX planes, respectively. Then the equation of the plane OLM is

(1) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ (2) $\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=0$ (3) $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0$ (4) $\frac{x}{a}-\frac{y}{b}+\frac{z}{c}=0$

52. The plane $\vec{r}\cdot\vec{n} = q$ will contain the line $\vec{r} = \vec{a} + \lambda\vec{b}$, if

(1) $\vec{b}\cdot\vec{n}\neq 0, \vec{a}\cdot\vec{n}\neq q$ (2) $\vec{b}\cdot\vec{n}= 0, \vec{a}\cdot\vec{n}\neq q$

(3) $\vec{b}\cdot\vec{n}= 0, \vec{a}\cdot\vec{n}= q$ (4) $\vec{b}\cdot\vec{n}\neq 0, \vec{a}\cdot\vec{n}= q$

53. The projection of point $P(\vec{p})$ on the plane $\vec{r}\cdot\vec{n} = q$ is $(\vec{s})$, then

(1) $\vec{s} = \frac{(q-\vec{p}\cdot\vec{n})\,\vec{n}}{|\vec{n}|^2}$ (2) $\vec{s} = \vec{p} + \frac{(q-\vec{p}\cdot\vec{n})\,\vec{n}}{|\vec{n}|^2}$

(3) $\vec{s} = \vec{p} - \frac{(\vec{p}\cdot\vec{n}\cdot)\,\vec{n}}{|\vec{n}|^2}$ (4) $\vec{s} = \vec{p} - \frac{(q-\vec{p}\cdot\vec{n})\,\vec{n}}{|\vec{n}|^2}$

54. The angle between $\hat{i}$ line of the intersection of the plane $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 0$ and $\vec{r}\cdot(3\hat{i}+3\hat{j}+\hat{k}) = 0$ is

(1) $\cos^{-1}\left(\frac{1}{3}\right)$ (2) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (3) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (4) none of these

55. The line $\frac{x+6}{5}=\frac{y+10}{3}=\frac{z+14}{8}$ is the hypotenuse of an isosceles right-angled triangle whose opposite vertex is (7, 2, 4). Then which of the following is not the side of the triangle?

(1) $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$ (2) $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$

(3) $\frac{x-7}{3}=\frac{y-2}{5}=\frac{z-4}{-1}$ (4) none of these

56. The equation of the plane which passes through the line of intersection of planes $\vec{r}\cdot\vec{n_1} = q_1, \vec{r}\cdot\vec{n_2} = q_2$ and is parallel to the line of intersection of planes $\vec{r}\cdot\vec{n_3} = q_3$ and $\vec{r}\cdot\vec{n_4} = q_4$ is

(1) $[\vec{n_2}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_1}-q_1) = [\vec{n_1}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_2}-q_2)$ (2) $[\vec{n_1}\,\vec{n_2}\,\vec{n_3}](\vec{r}\cdot\vec{n_4}-q_4) = [\vec{n_4}\,\vec{n_3}\,\vec{n_1}](\vec{r}\cdot\vec{n_2}-q_2)$

(3) $[\vec{n_4}\,\vec{n_3}\,\vec{n_1}](\vec{r}\cdot\vec{n_4}-q_4) = [\vec{n_1}\,\vec{n_2}\,\vec{n_3}](\vec{r}\cdot\vec{n_2}-q_2)$ (4) none of these

57. Consider triangle AOB in the x–y plane, where $A \equiv (1, 0, 0)$, $B \equiv (0, 2, 0)$, and $O \equiv (0, 0, 0)$. The new position of O, when triangle is rotated about side AB by 90° can be

(1) $\left(\frac{4}{5}, \frac{3}{5}, \frac{2}{\sqrt{5}}\right)$ (2) $\left(\frac{-3}{5}, \frac{\sqrt{2}}{5}, \frac{2}{\sqrt{5}}\right)$ (3) $\left(\frac{4}{5}, \frac{2}{5}, \frac{2}{\sqrt{5}}\right)$ (4) $\left(\frac{4}{5}, \frac{2}{5}, \frac{1}{\sqrt{5}}\right)$

58. Let $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = 2\hat{i} - \hat{k}$, then the point of intersection of the lines $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$ and $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$ is

(1) $(3, -1, 1)$ (2) $(3, 1, -1)$ (3) $(-3, 1, 1)$ (4) $(-3, -1, -1)$

59. The coordinates of the point P on the line $\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \lambda(-\hat{i} + \hat{j} - \hat{k})$ which is nearest to the origin is

(1) $\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$ (2) $\left(-\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right)$ (3) $\left(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}\right)$ (4) None of these

60. The ratio in which the line segment joining the points whose position vectors are $2\hat{i} - 4\hat{j} - 7\hat{k}$ and $-3\hat{i} + 5\hat{j} - 8\hat{k}$ is divided by the plane whose equation is $\hat{r} \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 13$ is

(1) 13 : 12 internally (2) 12 : 25 externally (3) 13 : 25 internally (4) 37 : 25 internally

61. Which of the following are equations for the plane passing through the points $P(1, 1, -1)$, $Q(3, 0, 2)$ and $R(-2, 1, 0)$?

(1) $(2\hat{i} - 3\hat{j} + 3\hat{k}) \cdot ((x+2)\hat{i} + (y-1)\hat{j} + z\hat{k}) = 0$

(2) $x = 3 - t, y = -11t, z = 2 - 3t$

(3) $(x + 2) + 11(y - 1) = 3z$

(4) $(2\hat{i} - \hat{j} + 3\hat{k}) \times (-3\hat{i} + \hat{k}) \cdot ((x+2)\hat{i} + (y-1)\hat{j} + z\hat{k}) = 0$

62. Given $\vec{\alpha} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{\beta} = \hat{i} - 2\hat{j} - 4\hat{k}$ are the position vectors of the points A and B. Then the distance of the point $-\hat{i} + \hat{j} + \hat{k}$ from the plane passing through B and perpendicular to AB is

(1) 5 (2) 10 (3) 15 (4) 20

63. L_1 and L_2 are two lines whose vector equations are

$L_1 : \vec{r} = \lambda((\cos\theta + \sqrt{3})\hat{i} + (\sqrt{2}\sin\theta)\hat{j} + (\cos\theta - \sqrt{3})\hat{k})$

$L_2 : \vec{r} = \mu(a\hat{i} + b\hat{j} + c\hat{k})$, where λ and μ are scalars and α is the acute angle between L_1 and L_2. If the angle α is independent of θ, then the value of α is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

64. The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

(1) $\sqrt{30}$ (2) $2\sqrt{30}$ (3) $5\sqrt{30}$ (4) $3\sqrt{30}$

65. The line through $\hat{i} + 3\hat{j} + 2\hat{k}$ and $\perp$ to the line $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + \hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$ is

(1) $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(-\hat{i} + 5\hat{j} - 3\hat{k})$ (2) $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(\hat{i} - 5\hat{j} + 3\hat{k})$

(3) $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(\hat{i} + 5\hat{j} + 3\hat{k})$ (4) $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(-\hat{i} - 5\hat{j} - 3\hat{k})$

66. The equation of the plane passing through the lines $\frac{x-4}{1} = \frac{y-3}{1} = \frac{z-2}{2}$ and $\frac{x-3}{1} = \frac{y-2}{-4} = \frac{z}{5}$ is

(1) $11x - y - 3z = 35$ (2) $11x + y - 3z = 35$ (3) $11x - y + 3z = 35$ (4) none of these

67. The three planes $4y + 6z = 5$, $2x + 3y + 5z = 5$ and $6x + 5y + 9z = 10$

(1) meet in a point (2) have a line in common

(3) form a triangular prism (4) none of these

68. The equation of the plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ and parallel to the line $y = 0$ and $z = 0$ is

(1) $(ab' - a'b)x + (bc' - b'c)y + (ad' - a'd) = 0$

(2) $(ab' - a'b)x + (bc' - b'c)y + (ad' - a'd)z = 0$

(3) $(ab' - a'b)y + (ac' - a'c)z + (ad' - a'd) = 0$

(4) none of these

69. Equation of the plane passing through the points (2, 2, 1) and (9, 3, 6), and perpendicular to the plane $2x + 6y + 6z - 1 = 0$ is

(1) $3x + 4y + 5z = 9$ (2) $3x + 4y - 5z = 9$ (3) $3x + 4y - 5z = 9$ (4) none of the above

70. Value of λ such that the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{\lambda}$ is perpendicular to normal to the plane $\vec{r} \cdot (2\vec{i} + 3\vec{j} + 4\vec{k}) = 0$ is

(1) $-\frac{13}{4}$ (2) $-\frac{17}{4}$ (3) 4 (4) none of these

71. The equation of the plane through the intersection of the planes $x + 2y + 3z - 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and passing through the origin is

(1) $17x + 14y + 11z = 0$ (2) $7x + 4y + z = 0$ (3) $x + 14y + 11z = 0$ (4) $17x + y + z = 0$

72. The plane $4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $5x + 3y + 10z = 25$. The equation of the plane in its new position is

(1) $x - 4y + 6z = 106$ (2) $x - 8y + 13z = 103$ (3) $x - 4y + 6z = 110$ (4) $x - 8y + 13z = 105$

73. The vector equation of the plane passing through the origin and the line of intersection of the planes $\vec{r} \cdot \vec{a} = \lambda$ and $\vec{r} \cdot \vec{b} = \mu$ is

(1) $\vec{r} \cdot (\lambda\vec{a} - \mu\vec{b}) = 0$ (2) $\vec{r} \cdot (\lambda\vec{b} - \mu\vec{a}) = 0$ (3) $\vec{r} \cdot (\lambda\vec{a} + \mu\vec{b}) = 0$ (4) $\vec{r} \cdot (\lambda\vec{b} + \mu\vec{a}) = 0$

74. The lines $\vec{r} = \vec{a} + \lambda(\vec{b} \times \vec{c})$ and $\vec{r} = \vec{b} + \mu(\vec{c} \times \vec{a})$ will intersect if

(1) $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$ (2) $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$ (3) $\vec{b} \times \vec{a} = \vec{c} \times \vec{a}$ (4) none of these

75. The projection of the line $\frac{x+1}{-1} = \frac{y}{2} = \frac{z-1}{3}$ on the plane $x - 2y + z = 6$ is the line of intersection of this plane with the plane

(1) $2x + y + 2 = 0$ (2) $3x + y - z = 2$ (3) $2x - 3y + 8z = 3$ (4) none of these

76. The direction cosines of a line satisfy the relations $\lambda(l + m) = n$ and $mn + nl + lm = 0$. The value of λ, for which the two lines are perpendicular to each other, is

(1) 1 (2) 2 (3) 1/2 (4) none of these

77. The intercepts made on the axes by the plane which bisects the line joining the points $(1, 2, 3)$ and $(-3, 4, 5)$ at right angles are

(1) $\left(-\frac{9}{2}, 9, 9\right)$ (2) $\left(\frac{9}{2}, 9, 9\right)$ (3) $\left(9, -\frac{9}{2}, 9\right)$ (4) $\left(9, \frac{9}{2}, 9\right)$

78. The pair of lines whose direction cosines are given by the equations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$ are

(1) parallel (2) perpendicular (3) inclined at $\cos^{-1}\left(\frac{1}{6}\right)$ (4) none of these

79. A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B and C. The locus of a centroid of the tetrahedron $OABC$ is

(1) $x^2 + y^2 + z^2 = 4k^2$ (2) $x^2 + y^2 + z^2 = k^2$ (3) $2(k^2 + y^2 + z)^2 = k^2$ (4) none of these

80. A plane passes through a fixed point (a, b, c). The locus of the foot of the perpendicular to it from the origin is a sphere of radius

(1) $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$ (2) $\sqrt{a^2 + b^2 + c^2}$ (3) $a^2 + b^2 + c^2$ (4) $\frac{1}{2}(a^2 + b^2 + c^2)$

81. What is the nature of the intersection of the set of planes $x + ay + (b + c)z + d = 0$, $x + by + (c + a)z + d = 0$ and $x + cy + (a + b)z + d = 0$?

(1) They meet at a point
(2) They form a triangular prism
(3) They pass through a line
(4) They are at equal distance from the origin

82. Find the equation of a straight line in the plane $\vec{r} \cdot \vec{n} = d$ which is parallel to $\vec{r} = \vec{a} + \lambda \vec{b}$ and passes through the foot of the perpendicular drawn from point $P(\vec{a})$ to $\vec{r} \cdot \vec{n} = d$ (where $\vec{n} \cdot \vec{b} = 0$).

(1) $\vec{r} = \vec{a} + \left(\frac{d - \vec{a} \cdot \vec{n}}{n^2}\right)\vec{n} + \lambda \vec{b}$ (2) $\vec{r} = \vec{a} + \left(\frac{d - \vec{a} \cdot \vec{n}}{n}\right)\vec{n} + \lambda \vec{b}$

(3) $\vec{r} = \vec{a} + \left(\frac{\vec{a} \cdot \vec{n} - d}{n^2}\right)\vec{n} + \lambda \vec{b}$ (4) $\vec{r} = \vec{a} + \left(\frac{\vec{a} \cdot \vec{n} - d}{n}\right)\vec{n} + \lambda \vec{b}$

83. What is the equation of the plane which passes through the z-axis and is perpendicular to the line $\frac{x - a}{\cos\theta} = \frac{y + 2}{\sin\theta} = \frac{z - 3}{0}$?

(1) $x + y \tan\theta = 0$ (2) $y + x \tan\theta = 0$ (3) $x\cos\theta - y\sin\theta = 0$ (4) $x\sin\theta - y\cos\theta = 0$

84. A straight line L on the xy-plane bisects the angle between OX and OY. What are the direction cosines of L?

(1) $< (1/\sqrt{2}), (1/\sqrt{2}), 0 >$
(2) $< (1/2), (\sqrt{3}/2), 0 >$
(3) $< 0, 0, 1 >$
(4) $< (2/3), (2/3), (1/3) >$

85. For what value(s) of a will the two points $(1, a, 1)$ and $(-3, 0, a)$ lie on opposite sides of the plane $3x + 4y - 12z + 13 = 0$?

(1) $a < -1$ or $a > 1/3$ (2) $a = 0$ only (3) $0 < a < 1$ (4) $-1 < a < 1$

86. If the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$ cuts the axes of coordinates at points A, B and C, then find the area of the triangle ABC.

(1) 18 sq unit (2) 36 sq unit (3) $3\sqrt{14}$ sq unit (4) $2\sqrt{14}$ sq unit

Multiple Correct Answers Type

1. Let PM be the perpendicular from the point $P(1, 2, 3)$ to the x–y plane. If $\overrightarrow{OP}$ makes an angle θ with the positive direction of the z-axis and $\overrightarrow{OM}$ makes an angle ϕ with the positive direction of x-axis, where O is the origin and θ and ϕ are acute angles, then

(1) $\cos\theta\cos\phi = 1/\sqrt{14}$ (2) $\sin\theta\sin\phi = 2/\sqrt{14}$ (3) $\tan\phi = 2$ (4) $\tan\theta = \sqrt{5}/3$

2. Let P_1 denote the equation of a plane to which the vector $(\hat{i}+\hat{j})$ is normal and which contains the line whose equation is $\vec{r} = \hat{i}+\hat{j}+\hat{k}+\lambda(\hat{i}-\hat{j}-\hat{k})$ and P_2 denote the equation of the plane containing the line L and a point with position vector $\hat{j}$. Which of the following holds good?

(1) The equation of P_1 is $x + y = 2$.

(2) The equation of P_2 is $\vec{r}\cdot(\hat{i}-2\hat{j}+\hat{k}) = 2$.

(3) The acute angle between P_1 and P_2 is $\cot^{-1}(\sqrt{3})$.

(4) The angle between the plane P_2 and the line L is $\tan^{-1}\sqrt{3}$.

3. If the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = q_1$, $\vec{r}\cdot(\hat{i}+2a\hat{j}+\hat{k}) = q_2$ and $\vec{r}\cdot(a\hat{i}+a^2\hat{j}+\hat{k}) = q_3$ intersect in a line, then the value of a is

(1) 1 (2) 1/2 (3) 2 (4) 0

4. A line with direction cosines proportional to 1, –5 and –2 meets lines $x = y + 5 = z + 1$ and $x + 5 = 3y = 2z$. The coordinates of each of the points of the intersection are given by

(1) (2, – 3, 1) (2) (1, 2, 3) (3) (0, 5/3, 5/2) (4) (3, – 2, 2)

5. Let $P = 0$ be the equation of a plane passing through the line of intersection of the planes $2x - y = 0$ and $3z - y = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$. Then the points which lie on the plane $P = 0$ is/are

(1) (0, 9, 17) (2) (1/7, 2, 1/9) (3) (1, 3, – 4) (4) (1/2, 1, 1/3)

6. The equation of the lines $x + y + z - 1 = 0$ and $4x + y - 2z + 2 = 0$ written in the symmetrical form is

(1) $\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-2}{2}$ (2) $\frac{x+(1/2)}{1} = \frac{y-1}{-2} = \frac{z-(1/2)}{1}$

(3) $\frac{x}{1} = \frac{y}{-2} = \frac{z-1}{1}$ (4) $\frac{x+1}{1} = \frac{y-2}{-2} = \frac{z-0}{1}$

7. Consider the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z = 3$. The plane $67x - 162y + 47z + 44 = 0$ bisects the angle between the given planes which

(1) contains origin (2) is acute (3) is obtuse (4) none of these

8. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{\lambda}$ and $\frac{x-1}{\lambda} = \frac{y-4}{2} = \frac{z-5}{1}$ intersect, then

(1) $\lambda = -1$ (2) $\lambda = 2$ (3) $\lambda = -3$ (4) $\lambda = 0$

9. The equations of the plane which passes through (0, 0, 0) and which is equally inclined to the planes $x - y + z - 3 = 0$ and $x + y + z + 4 = 0$ is/are

(1) $y = 0$ (2) $x = 0$ (3) $x + y = 0$ (4) $x + z = 0$

10. The x–y plane is rotated about its line of intersection with the y–z plane by 45°, then the equation of the new plane is/are

(1) $z+x=0$ (2) $z-y=0$ (3) $x+y+z=0$ (4) $z-x=0$

11. The equation of the plane which is equally inclined to the lines $\frac{x-1}{2}=\frac{y}{-2}=\frac{z+2}{-1}$ and $\frac{x+3}{8}=\frac{y-4}{1}=\frac{z}{-4}$ and passing through the origin is/are

(1) $14x-5y-7z=0$ (2) $2x+7y-z=0$ (3) $3x-4y-z=0$ (4) $x+2y-5z=0$

12. Which of the following lines lie on the plane $x+2y-z+4=0$?

(1) $\frac{x-1}{1}=\frac{y}{-1}=\frac{z-5}{-1}$ (2) $x-y+z=2x+y-z=0$

(3) $\vec{r}=2\hat{i}-\hat{j}+4\hat{k}+\lambda(3\hat{i}+\hat{j}+5\hat{k})$ (4) none of these

13. If the volume of tetrahedron $ABCD$ is 1 cubic units, where $A(0, 1, 2)$, $B(-1, 2, 1)$ and $C(1, 2, 1)$, then the locus of point D is

(1) $x+y-z=3$ (2) $y+z=6$ (3) $y+z=0$ (4) $y+z=-3$

14. A rod of length 2 units whose one end is $(1, 0, -1)$ and other end touches the plane $x-2y+2z+4=0$, then

(1) the rod sweeps the figure whose volume is π cubic units.

(2) the area of the region which the rod traces on the plane is 2π.

(3) the length of projection of the rod on the plane is $\sqrt{3}$ units.

(4) the centre of the region which the rod traces on the plane is $\left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$.

15. Consider a set of points R in the space which is at a distance of 2 units from the line $\frac{x}{1}=\frac{y-1}{-1}=\frac{z+2}{2}$ between the planes $x-y+2z+3=0$ and $x-y+2z-2=0$.

(1) The volume of the bounded figure by points R and the planes is $(10/3\sqrt{3})\pi$ cube units.

(2) The area of the curved surface formed by the set of points R is $(20\pi/\sqrt{6})$ sq. units.

(3) The volume of the bounded figure by the set of points R and the planes is $(20\pi/\sqrt{6})$ cubic units.

(4) The area of the curved surface formed by the set of points R is $(10/\sqrt{3})\pi$ sq. units.

16. The equation of a line passing through the point $\vec{a}$ parallel to the plane $\vec{r}.\vec{n}=q$ and perpendicular to the line $\vec{r}=\vec{b}+t\vec{c}$ is

(1) $\vec{r}=\vec{a}+\lambda(\vec{n}\times\vec{c})$ (2) $(\vec{r}-\vec{a})\times(\vec{n}\times\vec{c})=0$

(3) $\vec{r}=\vec{b}+\lambda(\vec{n}\times\vec{c})$ (4) none of these

17. The equation of the line $x+y+z-1=0$, $4x+y-2z+2=0$ written in the symmetrical form is

(1) $\frac{x+1}{1}=\frac{y-2}{-2}=\frac{z-0}{1}$ (2) $\frac{x}{1}=\frac{y}{-2}=\frac{z-1}{1}$

(3) $\frac{x+1/2}{1}=\frac{y-1}{-2}=\frac{z-1/2}{1}$ (4) $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{z-2}{2}$

Linked Comprehension Type

For Problems 1–3

Given four points $A(2, 1, 0)$, $B(1, 0, 1)$, $C(3, 0, 1)$ and $D(0, 0, 2)$. Point D lies on a line L orthogonal to the plane determined by the points A, B and C.

1. The equation of the plane ABC is
   (1) $x+y+z-3=0$ (2) $y+z-1=0$ (3) $x+z-1=0$ (4) $2y+z-1=0$
2. The equation of the line L is
   (1) $\vec{r}=2\hat{k}+\lambda(\hat{i}+\hat{k})$ (2) $\vec{r}=2\hat{k}+\lambda(2\hat{j}+\hat{k})$ (3) $\vec{r}=2\hat{k}+\lambda(\hat{j}+\hat{k})$ (4) none
3. The perpendicular distance of D from the plane ABC is
   (1) $\sqrt{2}$ (2) 1/2 (3) 2 (4) $1/\sqrt{2}$

For Problems 4–6

A ray of light comes along the line $L = 0$ and strikes the plane mirror kept along the plane $P = 0$ at B. $A(2, 1, 6)$ is a point on the line $L=0$ whose image about $P=0$ is A'. It is given that $L=0$ is $\frac{x-2}{3}=\frac{y-1}{4}=\frac{z-6}{5}$ and $P=0$ is $x+y-2z=3$.

4. The coordinates of A' are
   (1) (6, 5, 2) (2) (6, 5, –2) (3) (6, –5, 2) (4) none of these
5. The coordinates of B are
   (1) (5, 10, 6) (2) (10, 15, 11) (3) (–10, –15, –14) (4) none of these
6. If $L_1 = 0$ is the reflected ray, then its equation is
   (1) $\frac{x+10}{4}=\frac{y-5}{4}=\frac{z+2}{3}$ (2) $\frac{x+10}{3}=\frac{y+15}{5}=\frac{z+14}{5}$
   (3) $\frac{x+10}{4}=\frac{y+15}{5}=\frac{z+14}{3}$ (4) none of these

For Problems 7–9

Consider three planes $2x + py + 6z = 8$, $x + 2y + qz = 5$ and $x + y + 3z = 4$.

7. Three planes intersect at a point if
   (1) $p=2, q\neq 3$ (2) $p\neq 2, q\neq 3$ (3) $p\neq 2, q=3$ (4) $p=2, q=3$
8. Three planes do not have any common point of intersection if
   (1) $p=2, q\neq 3$ (2) $p\neq 2, q\neq 3$ (3) $p\neq 2, q=3$ (4) $p=2, q=3$
9. The planes have infinite points common among them if
   (1) $p=2, q\in 3$ (2) $p\in 2, q\in 3$ (3) $p\neq 2, q=3$ (4) $p=2, q=3$

For Problems 10–12

Consider a plane $x+y-z=1$ and point $A(1, 2, -3)$. A line L has the equation $x=1+3r$, $y=2-r$ and $z=3+4r$.

10. The coordinate of a point B of line L such that AB is parallel to the plane is
    (1) (10, –1, 15) (2) (–5, 4, –5) (3) (4, 1, 7) (4) (–8, 5, –9)
11. The equation of the plane containing line L and point A has the equation
    (1) $x-3y+5=0$ (2) $x+3y-7=0$ (3) $3x-y-1=0$ (4) $3x+y-5=0$
12. The distance between the points on the line which are at a distance of $4/\sqrt{3}$ from the plane is
    (1) $4\sqrt{26}$ (2) 20 (3) $10\sqrt{13}$ (4) none of these

Matrix Match Type

1. Match the following lists:

List I	List II
a. A vector perpendicular to the line $x = 2t + 1, y = t + 2$ and $z = -t - 3$	**p.** $7\hat{i} + 3\hat{j} + 5\hat{k}$
b. A vector parallel to the planes $x + y + z - 3 = 0$ and $2x - y + 3z = 0$	**q.** $4\hat{i} - \hat{j} - 3\hat{k}$
c. A vector along which the distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{-1}$ and $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} - 2\hat{k})$ is the shortest	**r.** $-11\hat{i} + 7\hat{j} + 5\hat{k}$
d. A vector normal to the plane $\vec{r} = -\hat{i} + 4\hat{j} - 6\hat{k} + \lambda(\hat{i} + 3\hat{j} - 2\hat{k}) + \mu(-\hat{i} + 2\hat{j} - 5\hat{k})$	**s.** $\hat{i} + 3\hat{j} + \hat{k}$

2. Match the following lists:

List I	List II
a. Lines $\frac{x-1}{-2} = \frac{y+2}{3} = \frac{z}{-1}$ and $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} + \hat{k})$ are	**p.** intersecting
b. Lines $\frac{x+5}{1} = \frac{y-3}{7} = \frac{z+3}{3}$ and $x - y + 2z - 4 = 0 = 2x + y - 3z + 5 = 0$ are	**q.** perpendicular
c. Lines $(x = t - 3, y = -2t + 1, z = -3t - 2)$ and $\vec{r} = (t+1)\hat{i} + (2t+3)\hat{j} + (-t-9)\hat{k}$ are	**r.** parallel
d. Lines $\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} - \hat{k})$ and $\vec{r} = (-\hat{i} - 2\hat{j} + 5\hat{k}) + s(\hat{i} - 2\hat{j} + \frac{3}{4}\hat{k})$ are	**s.** skew

3. Match the following lists:

List I	List II
a. The coordinates of a point on the line $x = 4y + 5, z = 3y - 6$ at a distance 3 from the point $(5, 3, -6)$ is/are	**p.** $(-1, -2, 0)$
b. The plane containing the lines $\frac{x-2}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and parallel to $\hat{i} + 4\hat{j} + 7\hat{k}$ has the point	**q.** $(5, 0, -6)$
c. A line passes through two points $A(2, -3, -1)$ and $B(8, -1, 2)$. The coordinates of a point on this line nearer to the origin and at a distance of 14 units from A is/are	**r.** $(2, 5, 7)$
d. The coordinates of the foot of the perpendicular from the point $(3, -1, 11)$ on the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is/are	**s.** $(14, 1, 5)$

4. Match the following lists:

List I	List II
a. The distance between the line $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and plane $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$	p. $\frac{25}{3\sqrt{14}}$
b. The distance between parallel planes $\vec{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 4$ and $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 9\hat{k}) + 13 = 0$ is	q. 13/7
c. The distance of a point (2, 5, – 3) from the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$ is	r. $\frac{10}{3\sqrt{3}}$
d. The distance of the point (1, 0, – 3) from the plane $x - y - z - 9 = 0$ measured parallel to line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$	s. 7

5. Match the following lists:

List I	List II
a. Image of the point (3, 5, 7) in the plane $2x + y + z = -18$ is	p. (– 1, – 1, –1)
b. The point of intersection of the line $\frac{x-2}{-3} = \frac{y-1}{-2} = \frac{z-3}{2}$ and the plane $2x + y - z = 3$ is	q. (–21, –7, –5)
c. The foot of the perpendicular from the point (1, 1, 2) to the plane $2x - 2y + 4z + 5 = 0$ is	r. $\left(\frac{5}{2}, \frac{2}{3}, \frac{8}{3}\right)$
d. The intersection point of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ is	s. $\left(-\frac{1}{12}, \frac{25}{12}, \frac{-2}{12}\right)$

Numerical Value Type

1. Find the distance of the z-axis from the image of the point $M(2, -3, 3)$ in the plane $x - 2y - z + 1 = 0$.
2. If the length of the projection of the line segment with points (1, 0, – 1) and (–1, 2, 2) to the plane $x + 3y - 5z = 6$ is d, then find the value of $[d/2]$, where $[\cdot]$ represent greatest integer function.
3. If the angle between the plane $x - 3y + 2z = 1$ and the line $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-1}{-3}$ is θ, then find the value of cosec θ.
4. Let the equation of the plane containing line $x - y - z - 4 = 0 = x + y + 2z - 4$ and parallel to the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$ be $x + Ay + Bz + C = 0$. Then find the value of $|A + B + C - 4|$.
5. Let $P(a, b, c)$ be any point on the plane $3x + 2y + z = 7$, then find the least value of $2(a^2 + b^2 + c^2)$.
6. The plane denoted by $P_1 : 4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $P_2 : 5x + 3y + 10z = 25$. If the plane in its new position is denoted by P, and the distance of this plane from the origin is d, then find the value of $[k/2]$ (where $[\cdot]$ represents greatest integer less than or equal to k).

7. The distance of the point $P(-2, 3, -4)$ from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$ is d, then find the value of $(2d - 8)$.
8. The position vectors of the four angular points of a tetrahedron $OABC$ are $(0, 0, 0)$, $(0, 0, 2)$, $(0, 4, 0)$ and $(6, 0, 0)$, respectively. A point P inside the tetrahedron is at the same distance r from the four plane faces of the tetrahedron. Find the value of $9r$.
9. The value of k such that $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$ is _____.
10. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then the value of k is _____.
11. A variable plane at a distance of 1 unit from the origin cuts the coordinate axes at A, B and C. If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$, then the value of k is _____.
12. A plane which is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$ passes through $(1, -2, 1)$. The square of distance of the plane from the point $(1, 2, 2)$ is _____.

Archives

JEE Main

Single Correct Answer Type

1. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lie in the plane $x + 3y - \alpha z + \beta = 0$. Then (α, β) equals

(1) $(6, -17)$ (2) $(-6, 7)$ (3) $(5, -15)$ (4) $(-5, 15)$

(AIEEE 2009)

2. The projections of a vector on the three coordinate axes are $6, -3, 2$, respectively. The direction cosines of the vector are

(1) $6, -3, 2$ (2) $\frac{6}{5}, -\frac{3}{5}, \frac{2}{5}$ (3) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$ (4) $-\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

(AIEEE 2009)

3. **Statement 1:** The point $A(3, 1, 6)$ is the mirror image of the point $B(1, 3, 4)$ in the plane $x - y + z = 5$.
Statement 2: The plane $x - y + z = 5$ bisects the line segment joining $A(3, 1, 6)$ and $B(1, 3, 4)$.

(1) Statement 1 is false, statement 2 is true.
(2) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
(3) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 2.
(4) Statement 1 is true, statement 2 is false.

(AIEEE 2010)

4. A line AB in three-dimensional space makes angles $45°$ and $120°$ with the positive x-axis and the positive y-axis, respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals

(1) $75°$ (2) $30°$ (3) $45°$ (4) $60°$

(AIEEE 2010)

5. **Statement 1:** The point $A(1, 0, 7)$ is the mirror image of the point $B(1, 6, 3)$ in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

Statement 2: The line: $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ bisects the line segment joining $A(1, 0, 7)$ and $B(1, 6, 3)$.

(1) Statement 1 is false, statement 2 is true.
(2) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
(3) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.
(4) Statement 1 is true, statement 2 is false. **(AIEEE 2011)**

6. If the angle between the line $x=\frac{y-1}{2}=\frac{z-3}{\lambda}$ and the plane $x+2y+3z=4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals

(1) $\frac{5}{3}$ (2) $\frac{2}{3}$ (3) $\frac{3}{2}$ (4) $\frac{2}{5}$ **(AIEEE 2011)**

7. An equation of a plane parallel to the plane $x-2y+2z-5=0$ and at a unit distance from the origin is

(1) $x-2y+2z-3=0$ (2) $x-2y+2z+1=0$
(3) $x-2y+2z-1=0$ (4) $x-2y+2z+5=0$ **(AIEEE 2012)**

8. If the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k is equal to

(1) -1 (2) $\frac{2}{9}$ (3) $\frac{9}{2}$ (4) 0 **(AIEEE 2012)**

9. Distance between two parallel planes $2x+y+2z=8$ and $4x+2y+4z+5=0$ is

(1) $\frac{3}{2}$ (2) $\frac{5}{2}$ (3) $\frac{7}{2}$ (4) $\frac{9}{2}$

(JEE Main 2013)

10. If the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, then k can have

(1) any value (2) exactly one value (3) exactly two values (4) exactly three values

(JEE Main 2013)

11. The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3=0$ is the line

(1) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (2) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$
(3) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ (4) $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$

(JEE Main 2014)

12. The equation of the plane containing the lines $2x-5y+z=3$, $x+y+4z=5$ and parallel to the plane $x+3y+6z=1$, is

(1) $2x+6y+12z=13$ (2) $x+3y+6z=-7$ (3) $x+3y+6z=7$ (4) $2x+6y+12z=-13$

(JEE Main 2015)

13. If the line, $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane, $lx + my - z = 9$, then $l^2 + m^2$ is equal to

(1) 18 (2) 5 (3) 2 (4) 26

(JEE Main 2016)

14. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along the line $x = y = z$ is

(1) $10\sqrt{3}$ (2) $\frac{10}{\sqrt{3}}$ (3) $\frac{20}{3}$ (4) $3\sqrt{10}$

(JEE Main 2016)

15. If the image of the point $P(1, -2, 3)$ in the plane, $2x + 3y - 4z + 22 = 0$ measured parallel to line $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ is Q, then PQ is equal to

(1) $6\sqrt{5}$ (2) $3\sqrt{5}$ (3) $2\sqrt{42}$ (4) $\sqrt{42}$

(JEE Main 2017)

16. The distance of the point $(1, 3, -7)$ from the plane passing through the point $(1, -1, -1)$, having normal perpendicular to both the lines $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$, is

(1) $\frac{10}{\sqrt{74}}$ (2) $\frac{20}{\sqrt{74}}$ (3) $\frac{10}{\sqrt{83}}$ (4) $\frac{5}{\sqrt{83}}$

(JEE Main 2017)

17. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$, then the distance of the origin from the plane, containing the lines L_1 and L_2, is

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{4\sqrt{2}}$ (3) $\frac{1}{3\sqrt{2}}$ (4) $\frac{1}{2\sqrt{2}}$

(JEE Main 2018)

18. The length of the projection of the line segment joining the points $(5, -1, 4)$ and $(4, -1, 3)$ on the plane $x + y + z = 7$ is

(1) $\sqrt{\frac{2}{3}}$ (2) $\frac{2}{\sqrt{3}}$ (3) $\frac{2}{3}$ (4) $\frac{1}{3}$

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. Let $P(3, 2, 6)$ be a point in space and Q be a point on line $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x - 4y + 3z = 1$ is

(1) 1/4 (2) −1/4 (3) 1/8 (4) −1/8

(IIT-JEE, 2009)

2. Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

(1) $x+2y-2z=0$ (2) $3x+2y-2z=0$ (3) $x-2y+z=0$ (4) $5x+2y-4z=0$

(IIT-JEE, 2010)

3. If the distance of the point $P(1,-2,1)$ from the plane $x+2y-2z=\alpha$, where $\alpha>0$, is 5, then the foot of the perpendicular from P to the place is

(1) $\left(\frac{8}{3},\frac{4}{3},-\frac{7}{3}\right)$ (2) $\left(\frac{4}{3},-\frac{4}{3},\frac{1}{3}\right)$ (3) $\left(\frac{1}{3},\frac{2}{3},\frac{10}{3}\right)$ (4) $\left(\frac{2}{3},-\frac{1}{3},\frac{5}{2}\right)$

(IIT-JEE, 2010)

4. The point P is the intersection of the straight line joining the points $Q(2,3,5)$ and $R(1,-1,4)$ with the plane $5x-4y-z=1$. If S is the foot of the perpendicular drawn from the point $T(2,1,4)$ to QR, then the length of the line segment PS is

(1) $\frac{1}{\sqrt{2}}$ (2) $\sqrt{2}$ (3) 2 (4) $2\sqrt{2}$

(IIT-JEE, 2012)

5. Perpendiculars are drawn from points on the line $\frac{x+2}{2}=\frac{y+1}{-1}=\frac{z}{3}$ to the plane $x+y+z=3$. The feet of perpendiculars lie on the line

(1) $\frac{x}{5}=\frac{y-1}{8}=\frac{z-2}{-13}$ (2) $\frac{x}{2}=\frac{y-1}{3}=\frac{z-2}{-5}$ (3) $\frac{x}{4}=\frac{y-1}{3}=\frac{z-2}{-7}$ (4) $\frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$

(JEE Advanced 2013)

6. Let P be the image of the point $(3,1,7)$ with respect to the plane $x-y+z=3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$

(1) $x+y-3z=0$ (2) $3x+z=0$ (3) $x-4y+7z=0$ (4) $2x-y=0$

(JEE Advanced 2016)

7. The equation of the plane passing through the point $(1,1,1)$ and perpendicular to the planes $2x+y-2z=5$ and $3x-6y-2z=7$, is

(1) $14x+2y+15z=31$ (2) $14x+2y-15z=1$ (3) $-14x+2y+15z=3$ (4) $14x-2y+15z=27$

(JEE Advanced 2017)

Multiple Correct Answers Type

1. Two lines $L_1: x=5, \frac{y}{3-\alpha}=\frac{z}{-2}$ and $L_2: x=\alpha, \frac{y}{-1}=\frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

(1) 1 (2) 2 (3) 3 (4) 4

(JEE Advanced 2013)

2. A line l passing through the origin is perpendicular to the lines

$l_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}, -\infty < t < \infty$, $l_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}, -\infty < s < \infty$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is(are)

(1) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (2) $(-1, -1, 0)$ (3) $(1, 1, 1)$ (4) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

(JEE Advanced 2013)

3. In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M?

(1) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (2) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$ (3) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (4) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

(JEE Advanced 2015)

4. In R^3, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point $(0, 1, 0)$ from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relations is (are) true?

(1) $2\alpha + \beta + 2\gamma + 2 = 0$ (2) $2\alpha - \beta + 2\gamma + 4 = 0$

(3) $2\alpha + \beta - 2\gamma - 10 = 0$ (4) $2\alpha - \beta + 2\gamma - 8 = 0$

(JEE Advanced 2015)

5. Consider a pyramid $OPQRS$ located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base $OPQR$ of the pyramid is a square with $OP = 3$. The point S is directly above the midpoint T of diagonal OQ such that $TS = 3$. Then

(1) the acute angle between OQ and OS is $\pi/3$

(2) the equation of the plane containing the triangle OQS is $x - y = 0$

(3) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(4) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

(JEE Advanced 2016)

6. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) **TRUE?**

(1) The line of intersection of P_1 and P_2 has direction ratios 1, 2, –1

(2) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 an P_2

(3) The acute angle between P_1 and P_2 is 60°

(4) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$

(JEE Advanced 2018)

Matrix Match Type

1. Consider the lines $L_1: \frac{x-1}{2}=\frac{y}{-1}=\frac{z+3}{1}$, $L_2: \frac{x-4}{1}=\frac{y+3}{1}=\frac{z+3}{2}$ and the planes $P_1: 7x+y+2z=3$, $P_2: 3x+5y-6z=4$. Let $ax+by+cz=d$ be the equation of the plane passing through the point of intersection of lines L_1 and L_2, and perpendicular to planes P_1 and P_2.

Match List I with List II.

List I	List II
a. $a=$	p. 13
b. $b=$	q. –3
c. $c=$	r. 1
d. $d=$	s. –2

(JEE Advanced 2013)

Numerical Value Type

1. If the distance between the plane $Ax-2y+z=d$ and the plane containing the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is $\sqrt{6}$, then find the value of $|d|$. **(IIT-JEE, 2010)**

2. Let P be a point in the first octant, whose image Q in the plane $x+y=3$ (that is, the line segment PQ is perpendicular to the plane $x+y=3$ and the mid-point of PQ lies in the plane $x+y=3$) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _____. **(JEE Advanced 2018)**

3. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where $O(0, 0, 0)$ is the origin. Let $S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p}=\overrightarrow{SP}$, $\vec{q}=\overrightarrow{SQ}$, $\vec{r}=\overrightarrow{SR}$ and $\vec{t}=\overrightarrow{ST}$, then the value of $|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})|$ is _____.

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2)	2. (3)	3. (1)	4. (3)	5. (2)	6. (3)	7. (3)	8. (1)	9. (3)	10. (1)
11. (1)	12. (2)	13. (3)	14. (2)	15. (2)	16. (1)	17. (2)	18. (4)	19. (4)	20. (4)
21. (3)	22. (1)	23. (2)	24. (2)	25. (1)	26. (3)	27. (3)	28. (2)	29. (3)	30. (3)
31. (4)	32. (4)	33. (1)	34. (4)	35. (4)	36. (2)	37. (4)	38. (3)	39. (3)	40. (4)
41. (4)	42. (2)	43. (2)	44. (1)	45. (3)	46. (1)	47. (2)	48. (2)	49. (3)	50. (3)
51. (2)	52. (3)	53. (2)	54. (4)	55. (3)	56. (1)	57. (3)	58. (2)	59. (1)	60. (2)

61. (4) **62.** (1) **63.** (1) **64.** (4) **65.** (2) **66.** (4) **67.** (2) **68.** (3) **69.** (2) **70.** (1)
71. (1) **72.** (1) **73.** (2) **74.** (2) **75.** (1) **76.** (2) **77.** (1) **78.** (3) **79.** (2) **80.** (1)
81. (3) **82.** (1) **83.** (1) **84.** (1) **85.** (1) **86.** (3)

Multiple Correct Answers Type

1. (2), (3), (4) **2.** (1), (3) **3.** (1), (2) **4.** (1), (2) **5.** (1), (4)
6. (2), (3), (4) **7.** (1), (2) **8.** (1), (4) **9.** (1), (3) **10.** (1), (4)
11. (1), (2) **12.** (1), (3) **13.** (2), (3) **14.** (1), (3), (4) **15.** (2), (3)
16. (1), (2) **17.** (1), (2), (3)

Linked Comprehension Type

1. (2) **2.** (3) **3.** (4) **4.** (2) **5.** (3) **6.** (3) **7.** (2) **8.** (3) **9.** (2) **10.** (4)
11. (2) **12.** (4)

Matrix Match Type

1. a → s; b → q; c → p; d → r
2. a → q, s; b → r; c → p, q; d → p
3. a → q; b → p; c → s; d → r
4. a → r; b → p; c → q; d → s
5. a → q; b → r; c → s; d → p

Numerical Value Type

1. (1) **2.** (3) **3.** (2) **4.** (6) **5.** (7) **6.** (7) **7.** (9) **8.** (6) **9.** (7) **10.** (45)
11. (9) **12.** (8)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2) **2.** (3) **3.** (3) **4.** (4) **5.** (3) **6.** (2) **7.** (1) **8.** (3) **9.** (3) **10.** (3)
11. (1) **12.** (3) **13.** (3) **14.** (1) **15.** (3) **16.** (3) **17.** (3) **18.** (1)

JEE Advanced

Single Correct Answer Type

1. (1) **2.** (3) **3.** (1) **4.** (1) **5.** (4) **6.** (3) **7.** (1)

Multiple Correct Answers Type

1. (1), (4) **2.** (2), (4) **3.** (1), (2) **4.** (2), (4) **5.** (2), (3), (4)
6. (3), (4)

Matrix Match Type

1. a → r; b → q; c → s; d → p

Numerical Value Type

1. (6) **2.** (8) **3.** (0.5)